

PCF: THE ADVANCED PCF THEOREMS

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ABSTRACT. This is a revised version of [Sh:430, §6].

Date: May 1, 2011.

The author thanks Alice Leonhardt for the beautiful typing. I thank Peter Komjath for some comments. Paper E69.

§ 1. ON PCF

This is a revised version of [Sh:430, §6] more self-contained, large part done according to lectures in the Hebrew University Fall 2003

Recall

Definition 1.1. Let $\bar{f} = \langle f_\alpha : \alpha < \delta \rangle$, $f_\alpha \in {}^\kappa \text{Ord}$, I an ideal on κ .

1) We say that $f \in {}^\kappa \text{Ord}$ is a \leq_I -l.u.b. of \bar{f} when:

- (a) $\alpha < \delta \Rightarrow f_\alpha \leq_I f$
- (b) if $f' \in {}^\kappa \text{Ord}$ and $(\forall \alpha < \delta)(f_\alpha \leq_I f')$ then $f \leq_I f'$.

2) We say that f is a \leq_I -e.u.b. of \bar{f} when

- (a) $\alpha < \delta \Rightarrow f_\alpha \leq_I f$
- (b) if $f' \in {}^\kappa \text{Ord}$ and $f' <_I \text{Max}\{f, 1_\kappa\}$ then $f' <_I \text{Max}\{f_\alpha, 1_\kappa\}$ for some $\alpha < \delta$.

3) \bar{f} is \leq_I -increasing if $\alpha < \beta \Rightarrow f_\alpha \leq_I f_\beta$, similarly $<_I$ -increasing. We say \bar{f} is eventually $<_I$ -increasing: it is \leq_I -increasing and $(\forall \alpha < \delta)(\exists \beta < \delta)(f_\alpha <_I f_\beta)$.

4) We may replace I by the dual ideal on κ .

Remark 1.2. For κ, I, \bar{f} as in Definition 1.1, if \bar{f} is a \leq_I -e.u.b. of \bar{f} then f is a \leq_I -l.u.b. of \bar{f} .

Definition 1.3. 1) We say that \bar{s} witness or exemplifies \bar{f} is $(< \sigma)$ -chaotic for D when, for some κ

- (a) $\bar{f} = \langle f_\alpha : \alpha < \delta \rangle$ is a sequence of members of ${}^\kappa \text{Ord}$
- (b) D is a filter on κ (or an ideal on κ)
- (c) \bar{f} is $<_D$ -increasing
- (d) $\bar{s} = \langle s_i : i < \kappa \rangle$, s_i a non-empty set of $< \sigma$ ordinals
- (e) for every $\alpha < \delta$ for some $\beta \in (\alpha, \delta)$ and $g \in \prod_{i < \kappa} s_i$ we have $f_\alpha \leq_D g \leq_D f_\beta$.

2) Instead “ $(< \sigma^+)$ -chaotic” we may say “ σ -chaotic”.

Claim 1.4. Assume

- (a) I an ideal on κ
- (b) $\bar{f} = \langle f_\alpha : \alpha < \delta \rangle$ is $<_I$ -increasing, $f_\alpha \in {}^\kappa \text{Ord}$
- (c) $J \supseteq I$ is an ideal on κ and \bar{s} witnesses \bar{f} is $(< \sigma)$ -chaotic for J .

Then \bar{f} has no \leq_I -e.u.b. f such that $\{i < \kappa : \text{cf}(f(i)) \geq \sigma\} \in J$.

Discussion 1.5. What is the aim of clause (c) of 1.4? For \leq_I -increasing sequence $\bar{f}, \langle f_\alpha : \alpha < \delta \rangle$ in ${}^\kappa \text{Ord}$ we are interested whether it has an appropriate \leq_I -e.u.b. Of course, I may be a maximal ideal on κ and $\langle f_t : t \in \text{cf}((\omega, <)^\kappa / D) \rangle$ is $<_I$ -increasing cofinal in $(\omega, <)^\kappa / D$, so it has an $<_I$ -e.u.b. the sequence $\omega_\kappa = \langle \omega : i < \kappa \rangle$, but this is not what interests us now; we like to have a \leq_I -e.u.b. g such that $(\forall i)(\text{cf}(g(i)) > \kappa)$.

Proof. Toward contradiction assume that $f \in {}^\kappa \text{Ord}$ is a \leq_I -e.u.b. of \bar{f} and $A_1 := \{i < \kappa : \text{cf}(f(i)) \geq \sigma\} \notin I$ hence $A \notin I$.

We define a function $f' \in {}^\kappa \text{Ord}$ as follows:

- ⊗ (a) if $i \in A$ then $f'(i) = \sup(s_i \cap f(i)) + 1$

(b) if $i \in \kappa \setminus A$ then $f'(i) = 0$.

Now that $i \in A \Rightarrow \text{cf}(g(i)) \geq \sigma > |s_i| \Rightarrow f'(i) < f(i) \leq \text{Max}\{g(i), 1\}$ and $i \in \kappa \setminus A \Rightarrow f'(i) = 0 \Rightarrow f'(i) < \text{Max}\{f(i), 1\}$. So by clause (b) of Definition 1.1(2) we know that for some $\alpha < \delta$ we have $f' <_I \text{Max}\{f_\alpha, 1\}$. But “ \bar{s} witness that \bar{f} is $(< \sigma)$ -chaotic” hence we can find $g \in \prod_{i < \kappa} s_i$ and $\beta \in (\alpha, \delta)$ such that

$f_\alpha \leq_I g \leq_I f_\beta$ and as \bar{f} is $<_I$ -increasing without loss of generality $g <_I f_\beta$.

So $A_2 := \{i < \kappa : f_\alpha(i) \leq g(i) < f_\beta(i) \leq f(i) \text{ and } f'(i) < \text{Max}\{f_\alpha(i), 1\} = \kappa\} \text{ mod } I$ hence $A := A_1 \cap A_2 \neq \emptyset \text{ mod } I$ hence $A \neq \emptyset$. So for any $i \in A$ we have $f_\alpha(i) \leq g(i) < f_\beta(i) \leq f(i)$ and $f(i) \in s_i$ hence $g(i) < f'(i) := \sup(s_i \cap f(i)) + 1$ and so $f'(i) \geq 1$.

Also $f'(i) < \text{Max}\{f_\alpha(i), 1\}$ hence $f'(i) < f_\alpha(i)$. Together $f'(i) < f_\alpha(i) \leq g(i) < f'(i)$, contradiction. $\square_{1.4}$

Lemma 1.6. Suppose $\text{cf}(\delta) > \kappa^+$, I an ideal on κ and $f_\alpha \in {}^\kappa \text{Ord}$ for $\alpha < \delta$ is \leq_I -increasing. Then there are $\bar{J}, \bar{s}, \bar{f}'$ satisfying:

- (A) $\bar{s} = \langle s_i : i < \kappa \rangle$, each s_i a set of $\leq \kappa$ ordinals,
- (B) $\sup\{f_\alpha(i) : \alpha < \delta\} \in s_i$; moreover is $\max(s_i)$
- (C) $\bar{f}' = \langle f'_\alpha : \alpha < \delta \rangle$ where $f'_\alpha \in \prod_{i < \kappa} s_i$ is defined by $f'_\alpha(i) = \text{Min}\{s_i \setminus f_\alpha(i)\}$,
(similar to rounding!)
- (D) $\text{cf}[f'_\alpha(i)] \leq \kappa$ (e.g. $f'_\alpha(i)$ is a successor ordinal) implies $f'_\alpha(i) = f_\alpha(i)$
- (E) $\bar{J} = \langle J_\alpha : \alpha < \delta \rangle$, J_α is an ideal on κ extending I (for $\alpha < \delta$), decreasing with α (in fact for some $a_{\alpha, \beta} \subseteq \kappa$ (for $\alpha < \beta < \kappa$) we have $a_{\alpha, \beta}/I$ decreases with β , increases with α and J_α is the ideal generated by $I \cup \{a_{\alpha, \beta} : \beta \text{ belongs to } (\alpha, \lambda)\}$) so possibly $J_\alpha = \mathcal{P}(\kappa)$ and possibly $J_\alpha = I$

such that:

- (F) if D is an ultrafilter on κ disjoint to J_α then f'_α/D is a $<_D$ -l.u.b and even $<_D$ -e.u.b. of $\langle f_\beta/D : \beta < \alpha \rangle$ which is eventually $<_D$ -increasing and $\{i < \kappa : \text{cf}[f'_\alpha(i)] > \kappa\} \in D$.

Moreover

- (F)⁺ if $\kappa \notin J_\alpha$ then f'_α is an $<_{J_\alpha}$ -e.u.b (= exact upper bound) of $\langle f_\beta : \beta < \delta \rangle$ and $\beta \in (\alpha, \delta) \Rightarrow f'_\beta =_{J_\alpha} f'_\alpha$
- (G) if D is an ultrafilter on κ disjoint to I but for every α not disjoint to J_α then \bar{s} exemplifies $\langle f_\alpha : \alpha < \delta \rangle$ is κ chaotic for D as exemplified by \bar{s} (see Definition 1.3), i.e., for some club E of δ , $\beta < \gamma \in E \Rightarrow f_\beta \leq_D f'_\beta <_D f_\gamma$
- (H) if $\text{cf}(\delta) > 2^\kappa$ then $\langle f_\alpha : \alpha < \delta \rangle$ has a \leq_I -l.u.b. and even \leq_I -e.u.b. and for every large enough α we have $I_\alpha = I$
- (I) if $b_\alpha =: \{i : f'_\alpha(i) \text{ has cofinality } \leq \kappa \text{ (e.g., is a successor)}\} \notin J_\alpha$ then: for every $\beta \in (\alpha, \delta)$ we have $f'_\alpha \upharpoonright b_\alpha = f'_\beta \upharpoonright b_\alpha \text{ mod } J_\alpha$.

Remark 1.7. Compare with [Sh:506].

Proof. Let $\alpha^* = \cup\{f_\alpha(i) + 1 : \alpha < \delta, i < \kappa\}$ and $S = \{j < \alpha^* : j \text{ has cofinality } \leq \kappa\}$, $\bar{e} = \langle e_j : j \in S \rangle$ be such that

- (a) $e_j \subseteq j, |e_j| \leq \kappa$ for every $j \in S$
- (b) if $j = i + 1$ then $e_j = \{i\}$
- (c) if j is limit, then $j = \sup(e_j)$ and $j' \in S \cap e_j \Rightarrow e_{j'} \subseteq e_j$.

For a set $a \subseteq \alpha^*$ let $cl_{\bar{e}}(a) = a \cup \bigcup_{j \in a \cap S} e_j$ hence by clause (c) clearly $cl_{\bar{e}}(cl_{\bar{e}}(a)) = cl_{\bar{e}}(a)$ and $[a \subseteq b \Rightarrow cl_{\bar{e}}(a) \subseteq cl_{\bar{e}}(b)]$ and $|cl_{\bar{e}}(a)| \leq |a| + \kappa$. We try to choose by induction on $\zeta < \kappa^+$, the following objects: $\alpha_\zeta, D_\zeta, g_\zeta, \bar{s}_\zeta = \langle s_{\zeta,i} : i < \kappa \rangle, \langle f_{\zeta,\alpha} : \alpha < \delta \rangle$ such that:

- ⊠ (a) $g_\zeta \in {}^\kappa \text{Ord}$ and $g_\zeta(i) \leq \bigcup \{f_\alpha(i) : \alpha < \delta\}$
- (b) $s_{\zeta,i} = cl_{\bar{e}}[\{g_\epsilon(i) : \epsilon < \zeta\} \cup \{\sup_{\alpha < \delta} f_\alpha(i)\}]$ so it is a set of $\leq \kappa$ ordinals increasing with ζ and $\sup_{\alpha < \delta} f_\alpha(i) \in s_{\zeta,i}$, moreover $\sup_{\alpha < \delta} f_\alpha(i) = \max(s_{\zeta,i})$
- (c) $f_{\zeta,\alpha} \in {}^\kappa \text{Ord}$ is defined by $f_{\zeta,\alpha}(i) = \text{Min}\{s_{\zeta,i} \setminus f_\alpha(i)\}$,
- (d) D_ζ is an ultrafilter on κ disjoint to I
- (e) $f_\alpha \leq_{D_\zeta} g_\zeta$ for $\alpha < \delta$
- (f) α_ζ is an ordinal $< \delta$
- (g) $\alpha_\zeta \leq \alpha < \delta \Rightarrow g_\zeta <_{D_\zeta} f_{\zeta,\alpha}$.

If we succeed, let $\alpha(*) = \sup\{\alpha_\zeta : \zeta < \kappa^+\}$, so as $\text{cf}(\delta) > \kappa^+$ clearly $\alpha(*) < \delta$. Now let $i < \kappa$ and look at $\langle f_{\zeta,\alpha(*)}(i) : \zeta < \kappa^+ \rangle$; by its definition (see clause (c)), $f_{\zeta,\alpha(*)}(i)$ is the minimal member of the set $s_{\zeta,i} \setminus f_{\alpha(*)}(i)$. This set increases with ζ , so $f_{\zeta,\alpha(*)}(i)$ decreases with ζ (though not necessarily strictly), hence is eventually constant; so for some $\xi_i < \kappa^+$ we have $\zeta \in [\xi_i, \kappa^+) \Rightarrow f_{\zeta,\alpha(*)}(i) = f_{\xi_i,\alpha(*)}(i)$. Let $\xi(*) = \sup_{i < \kappa} \xi_i$, so $\xi(*) < \kappa^+$, hence

$$\odot_1 \quad \zeta \in [\xi(*), \kappa^+) \text{ and } i < \kappa \Rightarrow f_{\zeta,\alpha(*)}(i) = f_{\xi(*),\alpha(*)}(i).$$

By clauses (e) + (g) of ⊠ we know that $f_{\alpha(*)} \leq_{D_{\xi(*)}} g_{\xi(*)} <_{D_{\xi(*)}} f_{\xi(*),\alpha(*)}$ hence for some $i < \kappa$ we have $f_{\alpha(*)}(i) \leq g_{\xi(*)}(i) < f_{\xi(*),\alpha(*)}(i)$. But $g_{\xi(*)}(i) \in s_{\xi(*)+1,i}$ by clause (b) of ⊠ hence recalling the definition of $f_{\xi(*)+1,\alpha(*)}(i)$ in clause (c) of ⊠ and the previous sentence $f_{\xi(*)+1,\alpha(*)}(i) \leq g_{\xi(*)}(i) < f_{\xi(*),\alpha(*)}(i)$, contradicting the statement \odot_1 .

So necessarily we are stuck in the induction process. Let $\zeta < \kappa^+$ be the first ordinal that breaks the induction. Clearly $s_{\zeta,i} (i < \kappa), f_{\zeta,\alpha} (\alpha < \delta)$ are well defined.

Let $s_i =: s_{\zeta,i}$ (for $i < \kappa$) and $f'_\alpha = f_{\zeta,\alpha}$ (for $\alpha < \delta$), as defined in ⊠, clearly they are well defined. Clearly s_i is a set of $\leq \kappa$ ordinals and:

- (*)₁ $f_\alpha \leq f'_\alpha$
- (*)₂ $\alpha < \beta \Rightarrow f'_\alpha \leq_I f'_\beta$
- (*)₃ if $b = \{i : f'_\alpha(i) < f'_\beta(i)\} \notin I$ and $\alpha < \beta < \delta$ then $f'_\alpha \restriction b <_I f'_\beta \restriction b$.

We let for $\alpha < \delta$

- \odot_2 $J_\alpha = \{b \subseteq \kappa : b \in I \text{ or } b \notin I \text{ and for every } \beta \in (\alpha, \delta) \text{ we have: } f'_\alpha \restriction (\kappa \setminus b) =_I f'_\beta \restriction (\kappa \setminus b)\}$
- \odot_3 for $\alpha < \beta < \delta$ we let $a_{\alpha,\beta} =: \{i < \kappa : f'_\alpha(i) < f'_\beta(i)\}$.

Then as $\langle f'_\alpha : \alpha < \delta \rangle$ is \leq_I -increasing (i.e., (*)₂):

- (*)₄ $a_{\alpha,\beta}/I$ increases with β , decreases with α , J_α increases with α
- (*)₅ J_α is an ideal on κ extending I , in fact is the ideal generated by $I \cup \{a_{\alpha,\beta} : \beta \in (\alpha, \delta)\}$
- (*)₆ if D is an ultrafilter on κ disjoint to J_α , then f'_α/D is a $<_D$ -lub of $\{f_\beta/D : \beta < \delta\}$.

[Why? We know that $\beta \in (\alpha, \delta) \Rightarrow a_{\alpha,\beta} = \emptyset \pmod D$, so $f_\beta \leq f'_\beta =_D f'_\alpha$ for $\beta \in (\alpha, \delta)$, so f'_α/D is an \leq_D -upper bound. If it is not a least upper bound then for some $g \in {}^\kappa\text{Ord}$, for every $\beta < \delta$ we have $f_\beta \leq_D g <_D f'_\alpha$ and we can get a contradiction to the choice of ζ, \bar{s}, f'_β because: (D, g, α) could serve as $D_\zeta, g_\zeta, \alpha_\zeta$.]

- (*)₇ If D is an ultrafilter on κ disjoint to I but not to J_α for every $\alpha < \delta$ then \bar{s} exemplifies that $\langle f_\alpha : \alpha < \delta \rangle$ is κ^+ -chaotic for D , see Definition 1.3.

[Why? For every $\alpha < \delta$ for some $\beta \in (\alpha, \delta)$ we have $a_{\alpha,\beta} \in D$, i.e., $\{i < \kappa : f'_\alpha(i) < f'_\beta(i)\} \in D$, so $\langle f'_\alpha/D : \alpha < \delta \rangle$ is not eventually constant, so if $\alpha < \beta$, $f'_\alpha <_D f'_\beta$ then $f'_\alpha <_D f_\beta$ (by (*)₃) and $f_\alpha \leq_D f'_\alpha$ (by (c)). So $f_\alpha \leq_D f'_\alpha <_D f_\beta$ as required.]

- (*)₈ if $\kappa \notin J_\alpha$ then f'_α is an \leq_{J_α} -e.u.b. of $\langle f_\beta : \beta < \delta \rangle$.

[Why? By (*)₆, f'_α is a \leq_{J_α} -upper bound of $\langle f_\beta : \beta < \delta \rangle$; so assume that it is not a \leq_{J_α} -e.u.b. of $\langle f_\beta : \beta < \delta \rangle$, hence there is a function g with domain κ , such that $g <_{J_\alpha} \text{Max}\{1, f'_\alpha\}$, but for no $\beta < \delta$ do we have

$$c_\beta =: \{i < \kappa : g(i) < \text{Max}\{1, f_\beta(i)\}\} = \kappa \pmod{J_\alpha}.$$

Clearly $\langle c_\beta : \beta < \delta \rangle$ is increasing modulo J_α so there is an ultrafilter D on κ disjoint to $J_\alpha \cup \{c_\beta : \beta < \delta\}$. So $\beta < \delta \Rightarrow f_\beta \leq_D g \leq_D f'_\alpha$, so we get a contradiction to (*)₆ except when $g =_D f'_\alpha$ and then $f'_\alpha =_D 0_\kappa$ (as $g(i) < 1 \vee g(i) < f'_\alpha(i)$). If we can demand $c^* = \{i : f'_\alpha(i) = 0\} \notin D$ we are done, but easily $c^* \setminus c_\beta \in J_\alpha$ so we finish.]

- (*)₉ If $\text{cf}[f'_\alpha(i)] \leq \kappa$ then $f'_\alpha(i) = f_\alpha(i)$ so clause (D) of the lemma holds.

[Why? By the definition of $s_\zeta = c\ell_{\bar{e}}[\dots]$ and the choice of \bar{e} , and of $f'_\alpha(i)$.]

- (*)₁₀ Clause (I) of the conclusion holds.

[Why? As $f_\alpha \leq_{J_\alpha} f_\beta \leq_{J_\alpha} f'_\alpha$ and $f_\alpha \upharpoonright b_\alpha =_{J_\alpha} f'_\alpha \upharpoonright b_\alpha$ by (*)₉.]

- (*)₁₁ if $\alpha < \beta < \delta$ then $f'_\alpha = f'_\beta \pmod{J_\alpha}$, so clause (F)⁺ holds.

[Why? First, \bar{f} is \leq_I -increasing hence it is \leq_{J_α} -increasing. Second, $\beta \leq \alpha \Rightarrow f_\beta \leq_I f_\alpha \leq f'_\alpha \Rightarrow f_\beta \leq_{J_\alpha} f'_\alpha$. Third, if $\beta \in (\alpha, \delta)$ then $a_{\alpha,\beta} = \{i < \kappa : f'_\alpha(i) < f'_\beta(i)\} \in J_\alpha$, hence $f'_\beta \leq_{J_\alpha} f'_\alpha$ but as $f_\alpha \leq_I f_\beta$ clearly $f'_\alpha \leq_I f'_\beta$ hence $f'_\alpha \leq_{J_\alpha} f'_\beta$, so together $f'_\alpha =_{J_\alpha} f'_\beta$.]

- (*)₁₂ if $\text{cf}(\delta) > 2^\kappa$ then for some $\alpha(*)$, $J_{\alpha(*)} = I$ (hence \bar{f} has a \leq_I -e.u.b.)

[Why? As $\langle J_\alpha : \alpha < \delta \rangle$ is a \subseteq -decreasing sequence of subsets of $\mathcal{P}(\kappa)$ it is eventually constant, say, i.e., there is $\alpha(*) < \delta$ such that $\alpha(*) \leq \alpha < \delta \Rightarrow J_\alpha = J_{\alpha(*)}$. Also $I \subseteq J_{\alpha(*)}$, but if $I \neq J_{\alpha(*)}$ then there is an ultrafilter D of κ disjoint to I but not to $J_{\alpha(*)}$ hence $\langle s_i : i < \kappa \rangle$ witness being κ -chaotic. But this implies $\text{cf}(\delta) \leq \prod_{i < \kappa} |s_i| \leq \kappa^\kappa = 2^\kappa$, contradiction.]

The reader can check the rest.

□_{1.6}

Example 1.8. 1) We show that l.u.b and e.u.b are not the same. Let I be an ideal on κ , $\kappa^+ < \lambda = \text{cf}(\lambda)$, $\bar{a} = \langle a_\alpha : \alpha < \lambda \rangle$ be a sequence of subsets of κ , (strictly) increasing modulo I , $\kappa \setminus a_\alpha \notin I$ but there is no $b \in \mathcal{P}(\kappa) \setminus I$ such that $\bigwedge_\alpha b \cap a_\alpha \in I$. [Does this occur? E.g., for $I = [\kappa]^{<\kappa}$, the existence of such \bar{a} is known to be consistent; e.g., MA and $\kappa = \aleph_0$ and $\lambda = 2^{\aleph_0}$. Moreover, for any κ and $\kappa^+ < \lambda = \text{cf}(\lambda) \leq 2^\kappa$ we can find $a_\alpha \subseteq \kappa$ for $\alpha < \lambda$ such that, e.g., any Boolean combination of the a_α 's has cardinality κ (less needed). Let I_0 be the ideal on κ generated by $[\kappa]^{<\kappa} \cup \{a_\alpha \setminus a_\beta : \alpha < \beta < \lambda\}$, and let I be maximal in $\{J : J \text{ an ideal on } \kappa, I_0 \subseteq J \text{ and } [\alpha < \beta < \lambda \Rightarrow a_\beta \setminus a_\alpha \notin J]\}$. So if G.C.H. fails, we have examples.]

For $\alpha < \lambda$, we let $f_\alpha : \kappa \rightarrow \text{Ord}$ be:

$$f_\alpha(i) = \begin{cases} \alpha & \text{if } i \in \kappa \setminus a_\alpha, \\ \lambda + \alpha & \text{if } i \in a_\alpha. \end{cases}$$

Now the constant function $f \in {}^\kappa \text{Ord}$, $f(i) = \lambda + \lambda$ is a l.u.b of $\langle f_\alpha : \alpha < \lambda \rangle$ but not an e.u.b. (both mod I) (no e.u.b. is exemplified by $g \in {}^\kappa \text{Ord}$ which is constantly λ).

2) Why do we require “ $\text{cf}(\delta) > \kappa^+$ ” rather than “ $\text{cf}(\delta) > \kappa$ ”? As we have to, by Kojman-Shelah [KjSh:673].

Recall (see [Sh:506, 2.3(2)])

Definition 1.9. We say that $\bar{f} = \langle f_\alpha : \alpha < \delta \rangle$ obeys $\langle u_\alpha : \alpha \in S \rangle$ when

- (a) $f_\alpha : w \rightarrow \text{Ord}$ for some fixed set w
- (b) S a set of ordinals
- (c) $u_\alpha \subseteq \alpha$
- (d) if $\alpha \in S \cap \delta$ and $\beta \in u_\alpha$ then $t \in w \Rightarrow f_\beta(t) \leq f_\alpha(t)$.

Claim 1.10. Assume I is an ideal on κ , $\bar{f} = \langle f_\alpha : \alpha < \delta \rangle$ is \leq_I -increasing and obeys $\bar{u} = \langle u_\alpha : \alpha \in S \rangle$. The sequence \bar{f} has a \leq_I -e.u.b. when for some S^+ we have \otimes_1 or \otimes_2 where

- \otimes_1 (a) $S^+ \subseteq \{\alpha < \delta : \text{cf}(\alpha) > \kappa\}$
- (b) S^+ is a stationary subset of δ
- (c) for each $\alpha \in S^+$ there are unbounded subsets u, v of α for which $\beta \in v \Rightarrow u \cap \beta \subseteq u_\beta$.
- \otimes_2 $S^+ = \{\delta\}$ and for δ clause (c) of \otimes_1 holds.

Proof. By [Sh:506]. □

Remark 1.11. 1) Connected to $\tilde{I}[\lambda]$, see [Sh:506].

Claim 1.12. Suppose J a σ -complete ideal on δ^* , $\mu > \kappa = \text{cf}(\mu)$, $\mu = \text{tlim}_J \langle \lambda_i : i < \delta \rangle$, $\delta^* < \mu$, $\lambda_i = \text{cf}(\lambda_i) > \delta^*$ for $i < \delta^*$ and $\lambda = \text{tcf}(\prod_{i < \delta^*} \lambda_i / J)$, and $\langle f_\alpha : \alpha < \lambda \rangle$ exemplifies this.

Then we have

- (*) if $\langle u_\beta : \beta < \lambda \rangle$ is a sequence of pairwise disjoint non-empty subsets of λ , each of cardinality $\leq \sigma$ (not $< \sigma!$) and $\alpha^* < \mu^+$, then we can find $B \subseteq \lambda$ such that:

- (a) $\text{otp}(B) = \alpha^*$,
- (b) if $\beta \in B, \gamma \in B$ and $\beta < \gamma$ then $\sup(u_\beta) < \min(u_\gamma)$,
- (c) we can find $s_\zeta \in J$ for $\zeta \in \bigcup_{i \in B} u_i$ such that: if $\zeta \in \bigcup_{\beta \in B} u_\beta, \xi \in \bigcup_{\beta \in B} u_\beta, \zeta < \xi$ and $i \in \delta \setminus (s_\zeta \cup s_\xi)$, then $f_\zeta(i) < f_\xi(i)$.

Proof. First assume $\alpha^* < \mu$. For each regular $\theta < \mu$, as $\theta^+ < \lambda = \text{cf}(\lambda)$ there is a stationary $S_\theta \subseteq \{\delta < \lambda : \text{cf}(\delta) = \theta < \delta\}$ which is in $\tilde{I}[\lambda]$ (see [Sh:420, 1.5]) which is equivalent (see [Sh:420, 1.2(1)]) to:

- (*) there is $\bar{C}^\theta = \langle C_\alpha^\theta : \alpha < \lambda \rangle$
 - (α) C_α^θ a subset of α , with no accumulation points (in C_α^θ),
 - (β) $[\alpha \in \text{nacc}(C_\beta^\theta) \Rightarrow C_\alpha^\theta = C_\beta^\theta \cap \alpha]$,
 - (γ) for some club E_θ^0 of λ ,
- $$[\delta \in S_\theta \cap E_\theta^0 \Rightarrow \text{cf}(\delta) = \theta < \delta \wedge \delta = \sup(C_\delta^\theta) \wedge \text{otp}(C_\delta^\theta) = \theta].$$

Without loss of generality $S_\theta \subseteq E_\theta^0$, and $\bigwedge_{\alpha < \delta} \text{otp}(C_\alpha^\theta) \leq \theta$. By [Sh:365, 2.3, Def.1.3]

for some club E_θ of λ , $\langle g_\ell(C_\alpha^\theta, E_\theta) : \alpha \in S_\theta \rangle$ guess clubs (i.e., for every club $E \subseteq E_\theta$ of λ , for stationarily many $\zeta \in S_\theta$, $g_\ell(C_\zeta^\theta, E_\theta) \subseteq E$) (remember $g_\ell(C_\delta^\theta, E_\theta) = \{\sup(\gamma \cap E_\theta) : \gamma \in C_\delta^\theta; \gamma > \text{Min}(E_\theta)\}$). Let $C_{\alpha^*}^{\theta,*} = \{\gamma \in C_\alpha^\theta : \gamma = \text{Min}(C_\alpha^\theta \setminus \sup(\gamma \cap E_\theta))\}$, they have all the properties of the C_α^θ 's and guess clubs in a weak sense: for every club E of λ for some $\alpha \in S_\theta \cap E$, if $\gamma_1 < \gamma_2$ are successive members of E then $|(\gamma_1, \gamma_2] \cap C_{\alpha^*}^{\theta,*}| \leq 1$; moreover, the function $\gamma \mapsto \sup(E \cap \gamma)$ is one to one on $C_{\alpha^*}^{\theta,*}$.

Now we define by induction on $\zeta < \lambda$, an ordinal α_ζ and functions $g_\theta^\zeta \in \prod_{i < \delta^*} \lambda_i$

(for each $\theta \in \Theta = \{\theta : \theta < \mu, \theta \text{ regular uncountable}\}$).

For given ζ , let $\alpha_\zeta < \lambda$ be minimal such that:

$$\xi < \zeta \Rightarrow \alpha_\xi < \alpha_\zeta$$

$$\xi < \zeta \text{ and } \theta \in \Theta \Rightarrow g_\theta^\xi < f_{\alpha_\zeta} \pmod{J}.$$

Now α_ζ exists as $\langle f_\alpha : \alpha < \lambda \rangle$ is $<_J$ -increasing cofinal in $\prod_{i < \delta^*} \lambda_i / J$. Now for each

$\theta \in \Theta$ we define g_θ^ζ as follows:

for $i < \delta^*$, $g_\theta^\zeta(i)$ is $\sup[\{g_\theta^\xi(i) + 1 : \xi \in C_\zeta^\theta\} \cup \{f_{\alpha_\zeta}(i) + 1\}]$ if this number is $< \lambda_i$, and $f_{\alpha_\zeta}(i) + 1$ otherwise.

Having made the definition we prove the assertion. We are given $\langle u_\beta : \beta < \lambda \rangle$, a sequence of pairwise disjoint non-empty subsets of λ , each of cardinality $\leq \sigma$ and $\alpha^* < \mu$. We should find B as promised; let $\theta = (|\alpha^*| + |\delta^*|)^+$ so $\theta < \mu$ is regular $> |\delta^*|$. Let $E = \{\delta \in E_\theta : (\forall \zeta)[\zeta < \delta \Leftrightarrow \sup(u_\zeta) < \delta \Leftrightarrow u_\zeta \subseteq \delta \Leftrightarrow \alpha_\zeta < \delta]\}$. Choose $\alpha \in S_\theta \cap \text{acc}(E)$ such that $g_\ell(C_\alpha^\theta, E_\theta) \subseteq E$; hence letting $C_{\alpha^*}^{\theta,*} = \{\gamma_i : i < \theta\}$ (increasing), $\gamma(i) = \gamma_i$, we know that $i < \delta^* \Rightarrow (\gamma_i, \gamma_{i+1}) \cap E \neq \emptyset$. Now let $B = \{\gamma_{5i+3} : i < \alpha^*\}$ we shall prove that B is as required. For $\alpha \in u_{\gamma(5\zeta+3)}, \zeta < \alpha^*$, let $s_\alpha^\zeta = \{i < \delta^* : g_\theta^{\gamma(5\zeta+1)}(i) < f_\alpha(i) < g_\theta^{\gamma(5\zeta+4)}(i)\}$, for each $\zeta < \alpha^*$ let $\langle \alpha_{\zeta, \varepsilon} : \varepsilon < |u_{\gamma(5\zeta+3)}| \rangle$ enumerate $u_{\gamma(5\zeta+3)}$ and let

$$s_{\alpha_{\zeta}, \epsilon}^1 = \{i : \text{for every } \xi < \epsilon, f_{\alpha_{\zeta}, \xi}(i) < f_{\alpha_{\zeta}, \epsilon}(i) \Leftrightarrow \alpha_{\zeta, \xi} < \alpha_{\zeta, \epsilon} \\ \Leftrightarrow f_{\alpha_{\zeta}, \xi}(i) \leq f_{\alpha_{\zeta}, \epsilon}(i)\}.$$

Lastly, for $\alpha \in \bigcup_{\zeta < \alpha^*} u_{5\zeta+3}$ let $s_\alpha = s_\alpha^0 \cup s_\alpha^1$ and it is enough to check that $\langle \zeta_\alpha : \alpha \in B \rangle$ witness that B is as required. Also we have to consider $\alpha^* \in [\mu, \mu^+)$, we prove this by induction on α^* and in the induction step we use $\theta = (\text{cf}(\alpha^*) + |\delta^*|)^+$ using a similar proof. $\square_{1.12}$

Remark 1.13. In 1.12:

1) We can avoid guessing clubs.

2) Assume $\sigma < \theta_1 < \theta_2 < \mu$ are regular and there is $S \subseteq \{\delta < \lambda : \text{cf}(\delta) = \theta_1\}$ from $I[\lambda]$ such that for every $\zeta < \lambda$ (or at least a club) of cofinality θ_2 , $S \cap \zeta$ is stationary and $\langle f_\alpha : \alpha < \lambda \rangle$ obey suitable \bar{C}^θ (see [Sh:345a, §2]). Then for some $A \subseteq \lambda$ unbounded, for every $\langle u_\beta : \beta < \theta_2 \rangle$ sequence of pairwise disjoint non-empty subsets of A , each of cardinality $< \sigma$ with $[\min u_\beta, \sup u_\beta]$ pairwise disjoint we have: for every $B_0 \subseteq A$ of order type θ_2 , for some $B \subseteq B_0$, $|B| = \theta_1$, (c) of (*) of 1.12 holds.

3) In (*) of 1.12, “ $\alpha^* < \mu$ ” can be replaced by “ $\alpha^* < \mu^+$ ” (prove by induction on α^*).

Observation 1.14. Assume $\lambda < \lambda^{<\lambda}$, $\mu = \text{Min}\{\tau : 2^\tau > \lambda\}$. *Then there are δ, χ and \mathcal{T} , satisfying the condition (*) below for $\chi = 2^\mu$ or at least arbitrarily large regular $\chi < 2^\mu$*

(*) \mathcal{T} a tree with δ levels, (where $\delta \leq \mu$) with a set X of $\geq \chi$ δ -branches, and for $\alpha < \delta$, $\bigcup_{\beta < \alpha} |\mathcal{T}_\beta| < \lambda$.

Proof. So let $\chi \leq 2^\mu$ be regular, $\chi > \lambda$.

Case 1: $\bigwedge_{\alpha < \mu} 2^{|\alpha|} < \lambda$. Then $\mathcal{T} = {}^{\mu>}2$, $\mathcal{T}_\alpha = {}^\alpha 2$ are O.K. (the set of branches ${}^\mu 2$ has cardinality 2^μ).

Case 2: Not Case 1. So for some $\theta < \mu$, $2^\theta \geq \lambda$, but by the choice of μ , $2^\theta \leq \lambda$, so $2^\theta = \lambda$, $\theta < \mu$ and so $\theta \leq \alpha < \mu \Rightarrow 2^{|\alpha|} = 2^\theta$. Note $|{}^{\mu>}2| = \lambda$ as $\mu \leq \lambda$. Note also that $\mu = \text{cf}(\mu)$ in this case (by the Bukovsky-Hechler theorem).

Subcase 2A: $\text{cf}(\lambda) \neq \mu = \text{cf}(\mu)$.

Let ${}^{\mu>}2 = \bigcup_{j < \lambda} B_j$, B_j increasing with j , $|B_j| < \lambda$. For each $\eta \in {}^\mu 2$, (as $\text{cf}(\lambda) \neq \text{cf}(\mu)$) for some $j_\eta < \lambda$,

$$\mu = \sup\{\zeta < \mu : \eta \restriction \zeta \in B_{j_\eta}\}.$$

So as $\text{cf}(\chi) \neq \mu$, for some ordinal $j^* < \lambda$ we have

$$\{\eta \in {}^\mu 2 : j_\eta \leq j^*\} \text{ has cardinality } \geq \chi.$$

As $\text{cf}(\lambda) \neq \text{cf}(\mu)$ and $\mu \leq \lambda$ (by its definition) clearly $\mu < \lambda$, hence $|B_{j^*}| \times \mu < \lambda$.

Let

$$\mathcal{T} = \{\eta \restriction \epsilon : \epsilon < \ell g(\eta) \text{ and } \eta \in B_{j^*}\}.$$

It is as required.

Subcase 2B: Not 2A so $\text{cf}(\lambda) = \mu = \text{cf}(\mu)$.

If $\lambda = \mu$ we get $\lambda = \lambda^{<\lambda}$ contradicting an assumption.

So $\lambda > \mu$, so λ singular. Now if $\alpha < \mu, \mu < \sigma_i = \text{cf}(\sigma_i) < \lambda$ for $i < \alpha$ then (see [Sh:345a, 1.3(10)]) $\max \text{pcf}\{\sigma_i : i < \alpha\} \leq \prod_{i < \alpha} \sigma_i \leq \lambda^{|\alpha|} \leq (2^\theta)^{|\alpha|} \leq 2^{<\mu} = \lambda$, but

as λ is singular and $\max \text{pcf}\{\sigma_i : i < \alpha\}$ is regular (see [Sh:345a, 1.9]), clearly the inequality is strict, i.e., $\max \text{pcf}\{\sigma_i : i < \alpha\} < \lambda$. So let $\langle \sigma_i : i < \mu \rangle$ be a strictly increasing sequence of regulars in (μ, λ) with limit λ , and by [Sh:355, 3.4] there is $\mathcal{T} \subseteq \prod_{i < \mu} \sigma_i$ satisfying $|\{\nu \restriction i : \nu \in \mathcal{T}\}| \leq \max \text{pcf}\{\sigma_j : j < i\} < \lambda$, and number of μ -branches $> \lambda$. In fact we can get any regular cardinal in $(\lambda, \text{pp}^+(\lambda))$ in the same way.

Let $\lambda^* = \min\{\lambda' : \mu < \lambda' \leq \lambda, \text{cf}(\lambda') = \mu \text{ and } \text{pp}(\lambda') > \lambda\}$, so (by [Sh:355, 2.3]), also λ^* has those properties and $\text{pp}(\lambda^*) \geq \text{pp}(\lambda)$. So if $\text{pp}^+(\lambda^*) = (2^\mu)^+$ or $\text{pp}(\lambda^*) = 2^\mu$ is singular, we are done. So assume this fails.

If $\mu > \aleph_0$, then (as in [Sh:430, 3.4]) $\alpha < 2^\mu \Rightarrow \text{cov}(\alpha, \mu^+, \mu^+, \mu) < 2^\mu$ and we can finish as in subcase 2A (actually $\text{cov}(2^{<\mu}, \mu^+, \mu^+, \mu) < 2^\mu$ suffices which holds by the previous sentence and [Sh:355, 5.4]). If $\mu = \aleph_0$ all is easy. $\square_{1.14}$

Claim 1.15. Assume $\mathfrak{b}_0 \subseteq \dots \subseteq \mathfrak{b}_k \subseteq \mathfrak{b}_{k+1} \subseteq \dots$ for $k < \omega$, $\mathfrak{a} = \bigcup_{k < \omega} \mathfrak{b}_k$ (and $|\mathfrak{a}|^+ < \text{Min}(\mathfrak{a})$) and $\lambda \in \text{pcf}(\mathfrak{a}) \setminus \bigcup_{k < \omega} \text{pcf}(\mathfrak{b}_k)$.

1) We can find finite $\mathfrak{d}_k \subseteq \text{pcf}(\mathfrak{b}_k \setminus \mathfrak{b}_{k-1})$ (stipulating $\mathfrak{b}_{-1} = \emptyset$) such that $\lambda \in \text{pcf}(\bigcup\{\mathfrak{d}_k : k < \omega\})$.

2) Moreover, we can demand $\mathfrak{d}_k \subseteq \text{pcf}(\mathfrak{b}_k) \setminus (\text{pcf}(\mathfrak{b}_{k-1}))$.

Proof. We start to repeat the proof of [Sh:371, 1.5] for $\kappa = \omega$. But there we apply [Sh:371, 1.4] to $\langle \mathfrak{b}_\zeta : \zeta < \kappa \rangle$ and get $\langle \langle \mathfrak{c}_{\zeta, \ell} : \ell \leq n(\zeta) \rangle : \zeta < \kappa \rangle$ and let $\lambda_{\zeta, \ell} = \max \text{pcf}(\mathfrak{c}_{\zeta, \ell})$. Here we apply the same claim ([Sh:371, 1.4]) to $\langle \mathfrak{b}_k \setminus \mathfrak{b}_{k-1} : k < \omega \rangle$ to get part (1). As for part (2), in the proof of [Sh:371, 1.5] we let $\delta = |\mathfrak{a}|^+ + \aleph_2$ choose $\langle N_i : i < \delta \rangle$, but now we have to adapt the proof of [Sh:371, 1.4] (applied to $\mathfrak{a}, \langle \mathfrak{b}_k : k < \omega \rangle, \langle N_i : i < \delta \rangle$); we have gotten there, toward the end, $\alpha < \delta$ such that $E_\alpha \subseteq E$. Let $E_\alpha = \{i_k : k < \omega\}, i_k < i_{k+1}$. But now instead of applying [Sh:371, 1.3] to each \mathfrak{b}_ℓ separately, we try to choose $\langle \mathfrak{c}_{\zeta, \ell} : \ell \leq n(\zeta) \rangle$ by induction on $\zeta < \omega$. For $\zeta = 0$ we apply [?, 1.3]. For $\zeta > 0$, we apply [Sh:371, 1.3] to \mathfrak{b}_ζ but there defining by induction on $\ell, \mathfrak{c}_\ell = \mathfrak{c}_{\zeta, \ell} \subseteq \mathfrak{a}$ such that $\max(\text{pcf}(\mathfrak{a} \setminus \mathfrak{c}_{\zeta, 0} \setminus \dots \setminus \mathfrak{c}_{\zeta, \ell-1}) \cap \text{pcf}(\mathfrak{b}_\zeta))$ is strictly decreasing with ℓ . \square

We use:

Observation 1.16. If $|\mathfrak{a}_i| < \text{Min}(\mathfrak{a}_i)$ for $i < i^*$, then $\mathfrak{c} = \bigcap_{i < i^*} \text{pcf}(\mathfrak{a}_i)$ has a last element or is empty.

Proof. By renaming without loss of generality $\langle |\mathfrak{a}_i| : i < i^* \rangle$ is non-decreasing. By [Sh:345b, 1.12]

$$(*)_1 \quad \mathfrak{d} \subseteq \mathfrak{c} \text{ and } |\mathfrak{d}| < \text{Min}(\mathfrak{d}) \Rightarrow \text{pcf}(\mathfrak{d}) \subseteq \mathfrak{c}.$$

By [Sh:371, 2.6] or 2.7(2)

(*)₂ if $\lambda \in \text{pcf}(\mathfrak{d}), \mathfrak{d} \subseteq \mathfrak{c}, |\mathfrak{d}| < \text{Min}(\mathfrak{d})$ then for some $\geq \subseteq \mathfrak{d}$ we have $|\geq| \leq \text{Min}(\mathfrak{a}_0), \lambda \in \text{pcf}(\geq)$.

Now choose by induction on $\zeta < |\mathfrak{a}_0|^+, \theta_\zeta \in \mathfrak{c}$, satisfying $\theta_\zeta > \max \text{pcf}\{\theta_\epsilon : \epsilon < \zeta\}$. If we are stuck in ζ , $\max \text{pcf}\{\theta_\epsilon : \epsilon < \zeta\}$ is the desired maximum by (*)₁. If we succeed the cardinal $\theta = \max \text{pcf}\{\theta_\epsilon : \epsilon < |\mathfrak{a}_0|^+\}$ is in $\text{pcf}\{\theta_\epsilon : \epsilon < \zeta\}$ for some $\zeta < |\mathfrak{a}_0|^+$ by (*)₂; easy contradiction. $\square_{1.16}$

Conclusion 1.17. Assume $\aleph_0 = \text{cf}(\mu) \leq \kappa \leq \mu_0 < \mu, [\mu' \in (\mu_0, \mu) \text{ and } \text{cf}(\mu') \leq \kappa \Rightarrow \text{pp}_\kappa(\mu') < \lambda]$ and $\text{pp}_\kappa^+(\mu) > \lambda = \text{cf}(\lambda) > \mu$. Then we can find λ_n for $n < \omega, \mu_0 < \lambda_n < \lambda_{n+1} < \mu, \mu = \bigcup_{n < \omega} \lambda_n$ and $\lambda = \text{tcf}(\prod_{n < \omega} \lambda_n / J)$ for some ideal J on ω (extending J_ω^{bd}).

Proof. Let $\mathfrak{a} \subseteq (\mu_0, \mu) \cap \text{Reg}, |\mathfrak{a}| \leq \kappa, \lambda \in \text{pcf}(\mathfrak{a})$. Without loss of generality $\lambda = \max \text{pcf}(\mathfrak{a})$, let $\mu = \bigcup_{n < \omega} \mu_n^0, \mu_0 \leq \mu_n^0 < \mu_{n+1}^0 < \mu$, let $\mu_n^1 = \mu_n^0 + \sup\{\text{pp}_\kappa(\mu') : \mu_0 < \mu' \leq \mu_n^0 \text{ and } \text{cf}(\mu') \leq \kappa\}$, by [Sh:355, 2.3] $\mu_n^1 < \mu, \mu_n^1 = \mu_n^0 + \sup\{\text{pp}_\kappa(\mu') : \mu_0 < \mu' < \mu_n^1 \text{ and } \text{cf}(\mu') \leq \kappa\}$ and obviously $\mu_n^1 \leq \mu_{n+1}^1$; by replacing by a subsequence without loss of generality $\mu_n^1 < \mu_{n+1}^1$. Now let $\mathfrak{b}_n = \mathfrak{a} \cap \mu_n^1$ and apply the previous claim 1.15: to $\mathfrak{b}_k =: \mathfrak{a} \cap (\mu_k^1)^+$, note:

$$\max \text{pcf}(\mathfrak{b}_k) \leq \mu_k^1 < \text{Min}(\mathfrak{b}_{k+1} \setminus \mathfrak{b}_k).$$

$\square_{1.17}$

Claim 1.18. 1) Assume $\aleph_0 < \text{cf}(\mu) = \kappa < \mu_0 < \mu, 2^\kappa < \mu$ and $[\mu_0 \leq \mu' < \mu \text{ and } \text{cf}(\mu') \leq \kappa \Rightarrow \text{pp}_\kappa(\mu') < \mu]$. If $\mu < \lambda = \text{cf}(\lambda) < \text{pp}^+(\mu)$ then there is a tree \mathcal{T} with κ levels, each level of cardinality $< \mu$, \mathcal{T} has exactly $\lambda \kappa$ -branches.

2) Suppose $\langle \lambda_i : i < \kappa \rangle$ is a strictly increasing sequence of regular cardinals, $2^\kappa < \lambda_0, \mathfrak{a} = \{\lambda_i : i < \kappa\}, \lambda = \max \text{pcf}(\mathfrak{a}), \lambda_j > \max \text{pcf}\{\lambda_i : i < j\}$ for each $j < \kappa$ (or at least $\sum_{i < j} \lambda_i > \max \text{pcf}\{\lambda_i : i < j\}$) and $\mathfrak{a} \notin J$ where $J = \{\mathfrak{b} \subseteq \mathfrak{a} : \mathfrak{b} \text{ is the union of countably many members of } J_{<\lambda}[\mathfrak{a}]\}$ (so $J \supseteq J_\mathfrak{a}^{\text{bd}}$ and $\text{cf}(\kappa) > \aleph_0$). Then the conclusion of (1) holds with $\mu = \sum_{i < \kappa} \lambda_i$.

Proof. 1) By (2) and [Sh:371, §1] (or can use the conclusion of [Sh:g, AG,5.7]).

2) For each $\mathfrak{b} \subseteq \mathfrak{a}$ define the function $g_\mathfrak{b} : \kappa \rightarrow \text{Reg}$ by

$$g_\mathfrak{b}(i) = \max \text{pcf}[\mathfrak{b} \cap \{\lambda_j : j < i\}].$$

Clearly $[\mathfrak{b}_1 \subseteq \mathfrak{b}_2 \Rightarrow g_{\mathfrak{b}_1} \leq g_{\mathfrak{b}_2}]$. As $\text{cf}(\kappa) > \aleph_0, J$ is \aleph_1 -complete, there is $\mathfrak{b} \subseteq \mathfrak{a}, \mathfrak{b} \notin J$ such that:

$$\mathfrak{c} \subseteq \mathfrak{b} \text{ and } \mathfrak{c} \notin J \Rightarrow \neg g_\mathfrak{c} <_J g_\mathfrak{b}.$$

Let $\lambda_i^* = \max \text{pcf}(\mathfrak{b} \cap \{\lambda_j : j < i\})$. For each i let $\mathfrak{b}_i = \mathfrak{b} \cap \{\lambda_j : j < i\}$ and $\langle \langle f_{\lambda, \alpha}^\mathfrak{b} : \alpha < \lambda \rangle : \lambda \in \text{pcf}(\mathfrak{b}) \rangle$ be as in [Sh:371, §1].

Let

$$\mathcal{T}_i^0 = \{ \text{Max}_{0 < \ell < n} f_{\lambda_\ell, \alpha_\ell}^\mathfrak{b} \upharpoonright \mathfrak{b}_i : \lambda_\ell \in \text{pcf}(\mathfrak{b}_i), \alpha_\ell < \lambda_\ell, n < \omega \}.$$

Let $\mathcal{T}_i = \{f \in \mathcal{T}_i^0 : \text{for every } j < i, f \restriction \mathbf{b}_j \in \mathcal{T}_j^0 \text{ moreover for some } f' \in \prod_{j < \kappa} \lambda_j, \text{ for every } j, f' \restriction \mathbf{b}_j \in \mathcal{T}_j^0 \text{ and } f \subseteq f'\}$, and $\mathcal{T} = \bigcup_{i < \kappa} \mathcal{T}_i$, clearly it is a tree, \mathcal{T}_i its i th level (or empty), $|\mathcal{T}_i| \leq \lambda_i^*$. By [Sh:371, 1.3,1.4] for every $g \in \prod \mathbf{b}$ for some $f \in \prod \mathbf{b}, \bigwedge_{i < \kappa} f \restriction \mathbf{b}_i \in \mathcal{T}_i^0$ hence $\bigwedge_{i < \kappa} f \restriction \mathbf{b}_i \in \mathcal{T}_i$. So $|\mathcal{T}_i| = \lambda_i^*$, and \mathcal{T} has $\geq \lambda \kappa$ -branches. By the observation below we can finish (apply it essentially to $\mathcal{F} = \{\eta : \text{for some } f \in \prod \mathbf{b} \text{ for } i < \kappa \text{ we have } \eta(i) = f \restriction \mathbf{b}_i \text{ and for every } i < \kappa, f \restriction \mathbf{b}_i \in \mathcal{T}_i^0\}$), then find $A \subseteq \kappa, \kappa \setminus A \in J$ and $g^* \in \prod_{i < \kappa} (\lambda_i + 1)$ such that $Y' =: \{f \in F : f \restriction A < g^* \restriction A\}$ has cardinality λ and then the tree will be \mathcal{T}' where $\mathcal{T}'_i =: \{f \restriction \mathbf{b}_i : f \in Y'\}$ and $\mathcal{T}' = \bigcup_{i < \kappa} \mathcal{T}'_i$. (So actually this proves that if we have such a tree with $\geq \theta(\text{cf}(\theta) > 2^\kappa)$ κ -branches then there is one with exactly θ κ -branches.) $\square_{1.18}$

Observation 1.19. *If $\mathcal{F} \subseteq \prod_{i < \kappa} \lambda_i$, J an \aleph_1 -complete ideal on κ , and $[f \neq g \in \mathcal{F} \Rightarrow f \neq_J g]$ and $|\mathcal{F}| \geq \theta, \text{cf}(\theta) > 2^\kappa$, then for some $g^* \in \prod_{i < \kappa} (\lambda_i + 1)$ we have:*

- (a) $Y = \{f \in \mathcal{F} : f <_J g^*\}$ has cardinality θ ,
- (b) for $f' <_J g^*$, we have $|\{f \in \mathcal{F} : f \leq_J f'\}| < \theta$,
- (c) there ¹ are $f_\alpha \in Y$ for $\alpha < \theta$ such that: $f_\alpha <_J g^*, [\alpha < \beta < \theta \Rightarrow \neg f_\beta <_J f_\alpha]$.

(Also in [Sh:829, §1]).

Proof. Let $Z =: \{g : g \in \prod_{i < \kappa} (\lambda_i + 1) \text{ and } Y_g =: \{f \in \mathcal{F} : f \leq_J g\} \text{ has cardinality } \geq \theta\}$. Clearly $\langle \lambda_i : i < \kappa \rangle \in Z$ so there is $g^* \in Z$ such that: $[g' \in Z \Rightarrow \neg g' <_J g^*]$; so clause (b) holds. Let $Y = \{f \in \mathcal{F} : f <_J g^*\}$, easily $Y \subseteq Y_{g^*}$ and $|Y_{g^*} \setminus Y| \leq 2^\kappa$ hence $|Y| \geq \theta$, also clearly $[f_1 \neq f_2 \in \mathcal{F} \text{ and } f_1 \leq_J f_2 \Rightarrow f_1 <_J f_2]$. If (a) fails, necessarily by the previous sentence $|Y| > \theta$. For each $f \in Y$ let $Y_f = \{h \in Y : h \leq_J f\}$, so by clause (b) we have $|Y_f| < \theta$ hence by the Hajnal free subset theorem for some $Z' \subseteq Z, |Z'| = \lambda^+$, and $f_1 \neq f_2 \in Z' \Rightarrow f_1 \notin Y_{f_2}$ so $[f_1 \neq f_2 \in Z' \Rightarrow \neg f_1 <_J f_2]$. But there is no such Z' of cardinality $> 2^\kappa$ ([Sh:111, 2.2,p.264]) so clause (a) holds. As for clause (c): choose $f_\alpha \in \mathcal{F}$ by induction on α , such that $f_\alpha \in Y \setminus \bigcup_{\beta < \alpha} Y_{f_\beta}$; it exists by cardinality considerations and $\langle f_\alpha : \alpha < \theta \rangle$ is as required (in (c)). $\square_{1.19}$

Observation 1.20. *Let $\kappa < \lambda$ be regular uncountable, $2^\kappa < \mu_i < \lambda$ (for $i < \kappa$), μ_i increasing in i . The following are equivalent:*

- (A) *there is $\mathcal{F} \subseteq {}^\kappa \lambda$ such that:*
 - (i) $|\mathcal{F}| = \lambda$,
 - (ii) $|\{f \restriction i : f \in \mathcal{F}\}| \leq \mu_i$,
 - (iii) $[f \neq g \in \mathcal{F} \Rightarrow f \neq_{J^{\text{bd}}} g]$;
- (B) *there be a sequence $\langle \lambda_i : i < \kappa \rangle$ such that:*
 - (i) $2^\kappa < \lambda_i = \text{cf}(\lambda_i) \leq \mu_i$,

¹Or straightening clause (i) see the proof of 1.20

- (ii) $\max \text{pcf}\{\lambda_i : i < \kappa\} = \lambda$,
- (iii) for $j < \kappa$, $\mu_j \geq \max \text{pcf}\{\lambda_i : i < j\}$;
- (C) there is an increasing sequence $\langle \mathfrak{a}_i : i < \kappa \rangle$ such that $\lambda \in \text{pcf}(\bigcup_{i < \kappa} \mathfrak{a}_i)$, $\text{pcf}(\mathfrak{a}_i) \subseteq \mu_i$ (so $\text{Min}(\bigcup_{i < \kappa} \mathfrak{a}_i) > |\bigcup_{i < \kappa} \mathfrak{a}_i|$).

Proof. (B) \Rightarrow (A): By [Sh:355, 3.4].

(A) \Rightarrow (B): If $(\forall \theta)[\theta \geq 2^\kappa \Rightarrow \theta^\kappa \leq \theta^+]$ we can directly prove (B) if for a club of $i < \kappa$, $\mu_i > \bigcup_{j < i} \mu_j$, and contradict (A) if this fails. Otherwise every normal filter D on κ is nice (see [Sh:386, §1]). Let \mathcal{F} exemplify (A).

Let $K = \{(D, g) : D \text{ a normal filter on } \kappa, g \in {}^\kappa(\lambda + 1), \lambda = |\{f \in \mathcal{F} : f <_D g\}|\}$. Clearly K is not empty (let g be constantly λ) so by [Sh:386] we can find $(D, g) \in K$ such that:

$$(*)_1 \text{ if } A \subseteq \kappa, A \neq \emptyset \text{ mod } D, g_1 <_{D+A} g \text{ then } \lambda > |\{f \in \mathcal{F} : f <_{D+A} g_1\}|.$$

Let $\mathcal{F}^* = \{f \in \mathcal{F} : f <_D g\}$, so (as in the proof of 1.18) $|\mathcal{F}^*| = \lambda$.

We claim:

$$(*)_2 \text{ if } h \in \mathcal{F}^* \text{ then } \{f \in \mathcal{F}^* : \neg h \leq_D f\} \text{ has cardinality } < \lambda.$$

[Why? Otherwise for some $h \in \mathcal{F}^*$, $\mathcal{F}' = \{f \in \mathcal{F}^* : \neg h \leq_D f\}$ has cardinality λ , for $A \subseteq \kappa$ let $\mathcal{F}'_A = \{f \in \mathcal{F}' : f \restriction A \leq h \restriction A\}$ so $\mathcal{F}' = \bigcup \{\mathcal{F}'_A : A \subseteq \kappa, A \neq \emptyset \text{ mod } D\}$, hence (recall that $2^\kappa < \lambda$) for some $A \subseteq \kappa$, $A \neq \emptyset \text{ mod } D$ and $|\mathcal{F}'_A| = \lambda$; now $(D + A, h)$ contradicts $(*)_1$].

By $(*)_2$ we can choose by induction on $\alpha < \lambda$, a function $f_\alpha \in F^*$ such that $\bigwedge_{\beta < \alpha} f_\beta <_D f_\alpha$. By [Sh:355, 1.2A(3)] $\langle f_\alpha : \alpha < \lambda \rangle$ has an e.u.b. f^* . Let $\lambda_i = \text{cf}(f^*(i))$, clearly $\{i < \kappa : \lambda_i \leq 2^\kappa\} = \emptyset \text{ mod } D$, so without loss of generality $\bigwedge_{i < \kappa} \text{cf}(f^*(i)) > 2^\kappa$ so λ_i is regular $\in (2^\kappa, \lambda]$, and $\lambda = \text{tcf}(\prod_{i < \kappa} \lambda_i / D)$. Let $J_i = \{A \subseteq i : \max \text{pcf}\{\lambda_j : j \in A\} \leq \mu_i\}$; so (remembering (ii) of (A)) we can find $h_i \in \prod_{j < i} f^*(j)$ such that:

$$(*)_3 \text{ if } \{j : j < i\} \notin J_i, \text{ then for every } f \in \mathcal{F}, f \restriction i <_{J_i} h_i.$$

Let $h \in \prod_{i < \kappa} f^*(i)$ be defined by:

$h(i) = \sup\{h_j(i) : j \in (i, \kappa) \text{ and } \{j : j < i\} \notin J_i\}$. As $\bigwedge_i \text{cf}[f^*(i)] > 2^\kappa$, clearly $h < f^*$ hence by the choice of f^* for some $\alpha(*) < \lambda$ we have: $h <_D f_{\alpha(*)}$ and let $A = \{i < \kappa : h(i) < f_{\alpha(*)}(i)\}$, so $A \in D$. Define λ'_i as follows: λ'_i is λ_i if $i \in A$, and is $(2^\kappa)^+$ if $i \in \kappa \setminus A$. Now $\langle \lambda'_i : i < \kappa \rangle$ is as required in (B).

(B) \Rightarrow (C): Straightforward.

(C) \Rightarrow (B): By [Sh:371, §1]. □_{1.20}

Claim 1.21. *If $\mathcal{F} \subseteq {}^\kappa \text{Ord}$, $2^\kappa < \theta = \text{cf}(\theta) \leq |\mathcal{F}|$ then we can find $g^* \in {}^\kappa \text{Ord}$ and a proper ideal I on κ and $A \subseteq \kappa$, $A \in I$ such that:*

- (a) $\prod_{i < \kappa} g^*(i)/I$ has true cofinality θ , and for each $i \in \kappa \setminus A$ we have $\text{cf}[g^*(i)] > 2^\kappa$,

- (b) for every $g \in {}^\kappa\text{Ord}$ satisfying $g \restriction A = g^* \restriction A$, $g \restriction (\kappa \setminus A) < g^* \restriction (\kappa \setminus A)$ we can find $f \in \mathcal{F}$ such that: $f \restriction A = g^* \restriction A$, $g \restriction (\kappa \setminus A) < f \restriction (\kappa \setminus A) < g^* \restriction (\kappa \setminus A)$.

Proof. As in [Sh:410, 3.7], proof of $(A) \Rightarrow (B)$. (In short let $f_\alpha \in \mathcal{F}$ for $\alpha < \theta$ be distinct, χ large enough, $\langle N_i : i < (2^\kappa)^+ \rangle$ as there, $\delta_i =: \sup(\theta \cap N_i)$, $g_i \in {}^\kappa\text{Ord}$, $g_i(\zeta) =: \text{Min}[N \cap \text{Ord} \setminus f_{\delta_i}(\zeta)]$, $A \subseteq \kappa$ and $S \subseteq \{i < (2^\kappa)^+ : \text{cf}(i) = \kappa^+\}$ stationary, $[i \in S \Rightarrow g_i = g^*]$, $[\zeta < \alpha \text{ and } i \in S \Rightarrow [f_{\delta_i}(\zeta) = g^*(\zeta) \equiv \zeta \in A]]$ and for some $i(*) < (2^\kappa)^+$, $g^* \in N_{i(*)}$, so $[\zeta \in \kappa \setminus A \Rightarrow \text{cf}(g^*(\zeta)) > 2^\kappa]$. $\square_{1.21}$

Claim 1.22. Suppose D is a σ -complete filter on $\theta = \text{cf}(\theta)$, κ an infinite cardinal, $\theta > |\alpha|^\kappa$ for $\alpha < \sigma$, and for each $\alpha < \theta$, $\bar{\beta} = \langle \beta_\epsilon^\alpha : \epsilon < \kappa \rangle$ is a sequence of ordinals. Then for every $X \subseteq \theta$, $X \neq \emptyset \pmod D$ there is $\langle \beta_\epsilon^* : \epsilon < \kappa \rangle$ (a sequence of ordinals) and $w \subseteq \kappa$ such that:

- (a) $\epsilon \in \kappa \setminus w \Rightarrow \sigma \leq \text{cf}(\beta_\epsilon^*) \leq \theta$,
- (b) if $\beta'_\epsilon \leq \beta_\epsilon^*$ and $[\epsilon \in w \equiv \beta'_\epsilon = \beta_\epsilon^*]$, then $\{\alpha \in X : \text{for every } \epsilon < \kappa \text{ we have } \beta'_\epsilon \leq \beta_\epsilon^* \leq \beta_\epsilon^* \text{ and } [\epsilon \in w \equiv \beta_\epsilon^\alpha = \beta_\epsilon^*]\} \neq \emptyset \pmod D$.

Proof. Essentially by the same proof as 1.21 (replacing δ_i by $\text{Min}\{\alpha \in X : \text{for every } Y \in N_i \cap D \text{ we have } \alpha \in Y\}$). See more [Sh:513, §6]. (See [Sh:620, §7]). $\square_{1.22}$

Remark 1.23. We can rephrase the conclusion as:

- (a) $B =: \{\alpha \in X : \text{if } \epsilon \in w \text{ then } \beta_\epsilon^\alpha = \beta_\epsilon^*, \text{ and: if } \epsilon \in \kappa \setminus w \text{ then } \beta_\epsilon^\alpha \text{ is } < \beta_\epsilon^* \text{ but } > \sup\{\beta_\zeta^* : \zeta < \epsilon, \beta_\zeta^\alpha < \beta_\zeta^*\}\} \neq \emptyset \pmod D$
- (b) If $\beta'_\epsilon < \beta_\epsilon^*$ for $\epsilon \in \kappa \setminus w$ then $\{\alpha \in B : \text{if } \epsilon \in \kappa \setminus w \text{ then } \beta_\epsilon^\alpha > \beta'_\epsilon\} \neq \emptyset \pmod D$
- (c) $\epsilon \in \kappa \setminus w \Rightarrow \text{cf}(\beta'_\epsilon) \text{ is } \leq \theta \text{ but } \geq \sigma$.

Remark 1.24. If $|\mathfrak{a}| < \min(\mathfrak{a})$, $\mathcal{F} \subseteq \Pi \mathfrak{a}$, $|\mathcal{F}| = \theta = \text{cf}(\theta) \notin \text{pcf}(\mathfrak{a})$ and even $\theta > \sigma = \sup(\theta^+ \cap \text{pcf}(\mathfrak{a}))$ then for some $g \in \Pi \mathfrak{a}$, the set $\{f \in \mathcal{F} : f < g\}$ is unbounded in θ (or use a σ -complete D as in 1.23). (This is as $\Pi \mathfrak{a} / J_{<\theta}[\mathfrak{a}]$ is $\min(\text{pcf}(\mathfrak{a}) \setminus \theta)$ -directed as the ideal $J_{<\theta}[\mathfrak{a}]$ is generated by $\leq \sigma$ sets; this is discussed in [Sh:513, §6].)

Remark 1.25. It is useful to note that 1.22 is useful to use [Sh:462, §4.5.14]: e.g., for if $n < \omega$, $\theta_0 < \theta_1 < \dots < \theta_n$, satisfying $(*)$ below, for any $\beta'_\epsilon \leq \beta_\epsilon^*$ satisfying $[\epsilon \in w \equiv \beta'_\epsilon < \beta_\epsilon^*]$ we can find $\alpha < \gamma$ in X such that:

$$\epsilon \in w \equiv \beta_\epsilon^\alpha = \beta_\epsilon^*,$$

$$\{\epsilon, \zeta\} \subseteq \kappa \setminus w \text{ and } \{\text{cf}(\beta_\epsilon^*), \text{cf}(\beta_\zeta^*)\} \subseteq [\theta_\ell, \theta_{\ell+1}) \text{ and } \ell \text{ even} \Rightarrow \beta_\epsilon^\alpha < \beta_\zeta^\gamma,$$

$$\{\epsilon, \zeta\} \subseteq \kappa \setminus w \text{ and } \{\text{cf}(\beta_\epsilon^*), \text{cf}(\beta_\zeta^*)\} \subseteq [\theta_\ell, \theta_{\ell+1}) \text{ and } \ell \text{ odd} \Rightarrow \beta_\epsilon^\gamma < \beta_\zeta^\alpha$$

where

- (*) (a) $\epsilon \in \kappa \setminus w \Rightarrow \text{cf}(\beta_\epsilon^*) \in [\theta_0, \theta_n)$, and
- (b) $\max \text{pcf}[\{\text{cf}(\beta_\epsilon^*) : \epsilon \in \kappa \setminus w\} \cap \theta_\ell] \leq \theta_\ell$ (which holds if $\theta_\ell = \sigma_\ell^+$, $\sigma_\ell^\kappa = \sigma_\ell$ for $\ell \in \{\ell, \dots, n\}$). \square

§ 2. NICE GENERATING SEQUENCES

Claim 2.1. *For any \mathfrak{a} , $|\mathfrak{a}| < \text{Min}(\mathfrak{a})$, we can find $\bar{\mathfrak{b}} = \langle \mathfrak{b}_\lambda : \lambda \in \mathfrak{a} \rangle$ such that:*

(α) $\bar{\mathfrak{b}}$ is a generating sequence, i.e.

$$\lambda \in \mathfrak{a} \Rightarrow J_{\leq \lambda}[\mathfrak{a}] = J_{< \lambda}[\mathfrak{a}] + \mathfrak{b}_\lambda,$$

(β) $\bar{\mathfrak{b}}$ is smooth, i.e., for $\theta < \lambda$ in \mathfrak{a} ,

$$\theta \in \mathfrak{b}_\lambda \Rightarrow \mathfrak{b}_\theta \subseteq \mathfrak{b}_\lambda,$$

(γ) $\bar{\mathfrak{b}}$ is closed, i.e., for $\lambda \in \mathfrak{a}$ we have $\mathfrak{b}_\lambda = \mathfrak{a} \cap \text{pcf}(\mathfrak{b}_\lambda)$.

Definition 2.2. 1) For a set a and set \mathfrak{a} of regular cardinals let $\text{Ch}_a^\mathfrak{a}$ be the function with domain $a \cap \mathfrak{a}$ defined by $\text{Ch}_a^\mathfrak{a}(\theta) = \sup(a \cap \theta)$.

2) We may write N instead of $|N|$, where N is a model (usually an elementary submodel of $(\mathcal{H}(\chi), \in, <_\chi^*)$ for some reasonable χ).

Observation 2.3. *If $\mathfrak{a} \subseteq a$ and $|a| < \text{Min}(\mathfrak{a})$ then $\text{ch}_a^\mathfrak{a} \in \Pi \mathfrak{a}$.*

Proof. Let $\langle \mathfrak{b}_\theta[\mathfrak{a}] : \theta \in \text{pcf}(\mathfrak{a}) \rangle$ be as in [Sh:371, 2.6] or Definition [Sh:506, 2.12]. For $\lambda \in \mathfrak{a}$, let $\bar{f}^{\mathfrak{a}, \lambda} = \langle f_\alpha^{\mathfrak{a}, \lambda} : \alpha < \lambda \rangle$ be a $<_{J_{< \lambda}[\mathfrak{a}]}$ -increasing cofinal sequence of members of $\prod \mathfrak{a}$, satisfying:

(*)₁ if $\delta < \lambda, |\mathfrak{a}| < \text{cf}(\delta) < \text{Min}(\mathfrak{a})$ and $\theta \in \mathfrak{a}$ then:

$$f_\delta^{\mathfrak{a}, \lambda}(\theta) = \text{Min} \left\{ \bigcup_{\alpha \in C} f_\alpha^{\mathfrak{a}, \lambda}(\theta) : C \text{ a club of } \delta \right\}$$

[exists by [Sh:345a, Def.3.3, (2)^b + Fact 3.4(1)]]].

Let $\chi = \beth_\omega(\sup(\mathfrak{a}))^+$ and κ satisfies $|\mathfrak{a}| < \kappa = \text{cf}(\kappa) < \text{Min}(\mathfrak{a})$ (without loss of generality there is such κ) and let $\bar{N} = \langle N_i : i < \kappa \rangle$ be an increasing continuous sequence of elementary submodels of $(\mathcal{H}(\chi), \in, <_\chi^*)$, $N_i \cap \kappa$ an ordinal, $\bar{N} \upharpoonright (i+1) \in N_{i+1}$, $\|N_i\| < \kappa$, and $\mathfrak{a}, \langle \bar{f}^{\mathfrak{a}, \lambda} : \lambda \in \mathfrak{a} \rangle$ and κ belong to N_0 . Let $N_\kappa = \bigcup_{i < \kappa} N_i$. Clearly by 2.3

(*)₂ $\text{Ch}_{N_i}^\mathfrak{a} \in \Pi \mathfrak{a}$ for $i \leq \kappa$.

Now for every $\lambda \in \mathfrak{a}$ the sequence $\langle \text{Ch}_{N_i}^\mathfrak{a}(\lambda) : i \leq \kappa \rangle$ is increasing continuous (note that $\lambda \in N_0 \subseteq N_i \subseteq N_{i+1}$ and $N_i, \lambda \in N_{i+1}$ hence $\sup(N_i \cap \lambda) \in N_{i+1} \cap \lambda$ hence $\text{Ch}_{N_i}^\mathfrak{a}(\lambda)$ is $< \sup(N_{i+1} \cap \lambda)$). Hence $\{\text{Ch}_{N_i}^\mathfrak{a}(\lambda) : i < \kappa\}$ is a club of $\text{Ch}_{N_\kappa}^\mathfrak{a}(\lambda)$; moreover, for every club E of κ the set $\{\text{Ch}_{N_i}^\mathfrak{a}(\lambda) : i \in E\}$ is a club of $\text{Ch}_{N_\kappa}^\mathfrak{a}(\lambda)$. Hence by (*), for every $\lambda \in \mathfrak{a}$, for some club E_λ of κ ,

(*)₃ (α) if $\theta \in \mathfrak{a}$ and $E \subseteq E_\lambda$ is a club of κ then $f_{\sup(N_\kappa \cap \lambda)}^{\mathfrak{a}, \lambda}(\theta) = \bigcup_{\alpha \in E} f_{\sup(N_\alpha \cap \lambda)}^{\mathfrak{a}, \lambda}(\theta)$

(β) $f_{\sup(N_\kappa \cap \lambda)}^{\mathfrak{a}, \lambda}(\theta) \in \text{cl}(\theta \cap N_\kappa)$, (i.e., the closure as a set of ordinals).

Let $E = \bigcap_{\lambda \in \mathfrak{a}} E_\lambda$, so E is a club of κ . For any $i < j < \kappa$ let

$$\mathfrak{b}_\lambda^{i,j} = \{\theta \in \mathfrak{a} : \text{Ch}_{N_i}^\mathfrak{a}(\theta) < f_{\sup(N_j \cap \lambda)}^{\mathfrak{a}, \lambda}(\theta)\}.$$

(*)₄ for $i < j < \kappa$ and $\lambda \in \mathfrak{a}$, we have:

- (α) $J_{\leq \lambda}[\mathfrak{a}] = J_{< \lambda}[\mathfrak{a}] + \mathfrak{b}_{\lambda}^{i,j}$ (hence $\mathfrak{b}_{\lambda}^{i,j} = \mathfrak{b}_{\lambda}[\bar{\mathfrak{a}}] \bmod J_{< \lambda}[\mathfrak{a}]$),
- (β) $\mathfrak{b}_{\lambda}^{i,j} \subseteq \lambda^+ \cap \mathfrak{a}$,
- (γ) $\langle \mathfrak{b}_{\lambda}^{i,j} : \lambda \in \mathfrak{a} \rangle \in N_{j+1}$,
- (δ) $f_{\sup(N_{\kappa} \cap \lambda)}^{\mathfrak{a}, \lambda} \leq \text{Ch}_{N_{\kappa}}^{\mathfrak{a}} = \langle \sup(N_{\kappa} \cap \theta) : \theta \in \mathfrak{a} \rangle$.

[Why?

Clause (α): First as $\text{Ch}_{N_i}^{\mathfrak{a}} \in \Pi \mathfrak{a}$ (by 2.3) there is $\gamma < \lambda$ such that $\text{Ch}_{N_i}^{\mathfrak{a}} <_{J=\lambda[\mathfrak{a}]} f_{\gamma}^{\mathfrak{a}, \lambda}$ and as $\mathfrak{a} \cup \{\mathfrak{a}, N_i\} \subseteq \text{Ch}_{N_{i+1}}^{\mathfrak{a}}$ clearly $\text{Ch}_{N_i}^{\mathfrak{a}} \in N_{i+1}$ hence without loss of generality $\gamma \in \lambda \cap N_{i+1}$ but $i+1 \leq j$ hence $N_{i+1} \subseteq N_j$ hence $\gamma \in N_j$ hence $\gamma < \sup(N_j \cap \lambda)$ hence $f_{\gamma}^{\mathfrak{a}, \lambda} <_{J=\lambda[\mathfrak{a}]} f_{\sup(N_j \cap \lambda)}^{\mathfrak{a}, \lambda}$. Together $\text{Ch}_{N_i}^{\mathfrak{a}} <_{J=\lambda[\mathfrak{a}]} f_{\sup(N_j \cap \lambda)}^{\mathfrak{a}, \lambda}$ hence by the definition of $\mathfrak{b}_{\lambda}^{i,j}$ we have $\mathfrak{a} \setminus \mathfrak{b}_{\lambda}^{i,j} \in J_{=\lambda}[\mathfrak{a}]$ hence $\lambda \notin \text{pcf}(\mathfrak{a} \setminus \mathfrak{b}_{\lambda}^{i,j})$ so $J_{\leq \lambda}[\mathfrak{a}] \subseteq J_{< \lambda}[\mathfrak{a}] + \mathfrak{b}_{\lambda}^{i,j}$.

Second, $(\Pi \mathfrak{a}, <_{J_{\leq \lambda}[\mathfrak{a}]})$ is λ^+ -directed hence there is $g \in \Pi \mathfrak{a}$ such that $\alpha < \lambda \Rightarrow f_{\alpha}^{\mathfrak{a}, \lambda} <_{J_{\leq \lambda}[\mathfrak{a}]} g$. As $f_{\alpha}^{\mathfrak{a}, \lambda} \in N_0$ without loss of generality $g \in N_0$ hence $g \in N_i$ so $g < \text{Ch}_{N_i}^{\mathfrak{a}}$. By the choice of g , $f_{\sup(N_j \cap \lambda)}^{\mathfrak{a}, \lambda} <_{J_{\leq \lambda}[\mathfrak{a}]} g$ so together $f_{\sup(N_j \cap \lambda)}^{\mathfrak{a}, \lambda} <_{J_{\leq \lambda}[\mathfrak{a}]} \text{Ch}_{N_i}^{\mathfrak{a}}$ hence $\mathfrak{b}_{\lambda}^{i,j} \in J_{\leq \lambda}[\mathfrak{a}]$. As $J_{< \lambda}[\mathfrak{a}] \subseteq J_{\leq \lambda}[\mathfrak{a}]$ clearly $J_{< \lambda}[\mathfrak{a}] + \mathfrak{b}_{\lambda}^{i,j} \subseteq J_{\leq \lambda}[\mathfrak{a}]$. Together we are done.

Clause (β): Because $\Pi(\mathfrak{a} \setminus \lambda^+)$ is λ^+ -directed we have $\theta \in \mathfrak{a} \setminus \lambda^+ \Rightarrow \{\theta\} \notin J_{\leq \lambda}[\mathfrak{a}]$.

Clause (γ): As $\text{Ch}_{N_i}^{\mathfrak{a}}, f_{\sup(N_j \cap \lambda)}^{\mathfrak{a}, \lambda}, \bar{f}$ belongs to N_{j+1} .

Clause (δ): For $\theta \in \mathfrak{a}(\subseteq N_0)$ we have $f_{\sup(N_{\kappa} \cap \lambda)}^{\mathfrak{a}, \lambda}(\theta) = \cup \{f_{\sup(N_{\varepsilon} \cap \lambda)}^{\mathfrak{a}, \lambda}(\theta) : \varepsilon \in E_{\lambda}\} \leq \sup(N_{\kappa} \cap \theta)$.

So we have proved (*)₄.]

- (*)₅ $\varepsilon(*) < \kappa$ when $\varepsilon(*) = \cup \{\varepsilon_{\lambda, \theta} : \theta < \lambda \text{ are from } \mathfrak{a}\}$ where $\varepsilon_{\lambda, \theta} = \text{Min}\{\varepsilon < \kappa : \text{if } f_{\sup(N_{\kappa} \cap \lambda)}^{\mathfrak{a}, \lambda}(\theta) < \sup(N_{\kappa} \cap \theta) \text{ then } f_{\sup(N_{\varepsilon} \cap \lambda)}^{\mathfrak{a}, \lambda}(\theta) < \sup(N_{\varepsilon} \cap \theta)\}$.

[Why? Obvious.]

- (*)₆ $f_{\sup(N_{\kappa} \cap \lambda)}^{\mathfrak{a}, \lambda} \upharpoonright \mathfrak{b}_{\lambda}^{i,j} = \text{Ch}_{N_{\kappa}}^{\mathfrak{a}} \upharpoonright \mathfrak{b}_{\lambda}^{i,j}$ when $i < j$ are from $E \setminus \varepsilon(*)$.

[Why? Let $\theta \in \mathfrak{b}_{\lambda}^{i,j}$, so by (*)₃(β) we know that $f_{\sup(N_{\kappa} \cap \lambda)}^{\mathfrak{a}, \lambda}(\theta) \leq \text{Ch}_{N_{\kappa}}^{\mathfrak{a}}(\theta)$. If the inequality is strict then there is $\beta \in N_{\kappa} \cap \theta$ such that $f_{\sup(N_{\kappa} \cap \lambda)}^{\mathfrak{a}, \lambda}(\theta) \leq \beta < \text{Ch}_{N_{\kappa}}^{\mathfrak{a}}(\theta)$ hence for some $\varepsilon < \kappa$, $\beta \in N_{\varepsilon}$ hence $\zeta \in (\varepsilon, \kappa) \Rightarrow f_{\sup(N_{\kappa} \cap \lambda)}^{\mathfrak{a}, \lambda}(\theta) < \text{Ch}_{N_{\zeta}}^{\mathfrak{a}}(\theta)$ hence (as “ $i \geq \varepsilon_{\lambda, \theta}$ ” holds) we have $f_{\sup(N_{\kappa} \cap \lambda)}^{\mathfrak{a}, \lambda}(\theta) < \text{Ch}_{N_i}^{\mathfrak{a}}(\theta)$ so $f_{\sup(N_j \cap \lambda)}^{\mathfrak{a}, \lambda}(\theta) \leq f_{\sup(N_{\kappa} \cap \lambda)}^{\mathfrak{a}, \lambda}(\theta) < \text{Ch}_{N_i}^{\mathfrak{a}}(\theta)$, (the first inequality holds as $j \in E_{\lambda}$). But by the definition of $\mathfrak{b}_{\lambda}^{i,j}$ this contradicts $\theta \in \mathfrak{b}_{\lambda}^{i,j}$.]

We now define by induction on $\epsilon < |\mathfrak{a}|^+$, for $\lambda \in \mathfrak{a}$ (and $i < j < \kappa$), the set $\mathfrak{b}_{\lambda}^{i,j, \epsilon}$:

- (*)₇ (α) $\mathfrak{b}_{\lambda}^{i,j,0} = \mathfrak{b}_{\lambda}^{i,j}$
- (β) $\mathfrak{b}_{\lambda}^{i,j, \epsilon+1} = \mathfrak{b}_{\lambda}^{i,j, \epsilon} \cup \cup \{\mathfrak{b}_{\theta}^{i,j, \epsilon} : \theta \in \mathfrak{b}_{\lambda}^{i,j, \epsilon}\} \cup \{\theta \in \mathfrak{a} : \theta \in \text{pcf}(\mathfrak{b}_{\lambda}^{i,j, \epsilon})\}$,
- (γ) $\mathfrak{b}_{\lambda}^{i,j, \epsilon} = \bigcup_{\zeta < \epsilon} \mathfrak{b}_{\lambda}^{i,j, \zeta}$ for $\epsilon < |\mathfrak{a}|^+$ limit.

Clearly for $\lambda \in \mathfrak{a}$, $\langle \mathfrak{b}_\lambda^{i,j,\epsilon} : \epsilon < |\mathfrak{a}|^+ \rangle$ belongs to N_{j+1} and is a non-decreasing sequence of subsets of \mathfrak{a} , hence for some $\epsilon(i, j, \lambda) < |\mathfrak{a}|^+$, we have

$$[\epsilon \in (\epsilon(i, j, \lambda), |\mathfrak{a}|^+) \Rightarrow \mathfrak{b}_\lambda^{i,j,\epsilon} = \mathfrak{b}_\lambda^{i,j,\epsilon(i,j,\lambda)}].$$

So letting $\epsilon(i, j) = \sup_{\lambda \in \mathfrak{a}} \epsilon(i, j, \lambda) < |\mathfrak{a}|^+$ we have:

$$(*)_8 \quad \epsilon(i, j) \leq \epsilon < |\mathfrak{a}|^+ \Rightarrow \bigwedge_{\lambda \in \mathfrak{a}} \mathfrak{b}_\lambda^{i,j,\epsilon(i,j)} = \mathfrak{b}_\lambda^{i,j,\epsilon}.$$

We restrict ourselves to the case $i < j$ are from $E \setminus \varepsilon(*)$. Which of the properties required from $\langle \mathfrak{b}_\lambda : \lambda \in \mathfrak{a} \rangle$ are satisfied by $\langle \mathfrak{b}_\lambda^{i,j,\epsilon(i,j)} : \lambda \in \mathfrak{a} \rangle$? In the conclusion of 2.1 properties (β) , (γ) hold by the inductive definition of $\mathfrak{b}_\lambda^{i,j,\epsilon}$ (and the choice of $\epsilon(i, j)$). As for property (α) , one half, $J_{\leq \lambda}[\mathfrak{a}] \subseteq J_{< \lambda}[\mathfrak{a}] + \mathfrak{b}_\lambda^{i,j,\epsilon(i,j)}$ hold by $(*)_4(\alpha)$ (and $\mathfrak{b}_\lambda^{i,j} = \mathfrak{b}_\lambda^{i,j,0} \subseteq \mathfrak{b}_\lambda^{i,j,\epsilon(i,j)}$), so it is enough to prove (for $\lambda \in \mathfrak{a}$):

$$(*)_9 \quad \mathfrak{b}_\lambda^{i,j,\epsilon(i,j)} \in J_{\leq \lambda}[\mathfrak{a}].$$

For this end we define by induction on $\epsilon < |\mathfrak{a}|^+$ functions $f_\alpha^{a,\lambda,\epsilon}$ with domain $\mathfrak{b}_\lambda^{i,j,\epsilon}$ for every pair (α, λ) satisfying $\alpha < \lambda \in \mathfrak{a}$, such that $\zeta < \epsilon \Rightarrow f_\alpha^{a,\lambda,\zeta} \subseteq f_\alpha^{a,\lambda,\epsilon}$, so the domain increases with ϵ .

We let $f_\alpha^{a,\lambda,0} = f_\alpha^{a,\lambda} \upharpoonright \mathfrak{b}_\lambda^{i,j}$, $f_\alpha^{a,\lambda,\epsilon} = \bigcup_{\zeta < \epsilon} f_\alpha^{a,\lambda,\zeta}$ for limit $\epsilon < |\mathfrak{a}|^+$ and $f_\alpha^{a,\lambda,\epsilon+1}$ is

defined by defining each $f_\alpha^{a,\lambda,\epsilon+1}(\theta)$ as follows:

Case 1: If $\theta \in \mathfrak{b}_\lambda^{i,j,\epsilon}$ then $f_\alpha^{a,\lambda,\epsilon+1}(\theta) = f_\alpha^{a,\lambda,\epsilon}(\theta)$.

Case 2: If $\mu \in \mathfrak{b}_\lambda^{i,j,\epsilon}$, $\theta \in \mathfrak{b}_\mu^{i,j,\epsilon}$ and not Case 1 and μ minimal under those conditions, then $f_\alpha^{a,\lambda,\epsilon+1}(\theta) = f_\beta^{a,\mu,\epsilon}(\theta)$ where we choose $\beta = f_\alpha^{a,\lambda,\epsilon}(\mu)$.

Case 3: If $\theta \in \mathfrak{a} \cap \text{pcf}(\mathfrak{b}_\lambda^{i,j,\epsilon})$ and neither Case 1 nor Case 2, then

$$f_\alpha^{a,\lambda,\epsilon+1}(\theta) = \text{Min}\{\gamma < \theta : f_\alpha^{a,\lambda,\epsilon} \upharpoonright \mathfrak{b}_\theta[\mathfrak{a}] \leq_{J_{< \theta}[\mathfrak{a}]} f_\gamma^{a,\theta,\epsilon}\}.$$

Now $\langle \langle \mathfrak{b}_\lambda^{i,j,\epsilon} : \lambda \in \mathfrak{a} \rangle : \epsilon < |\mathfrak{a}|^+ \rangle$ can be computed from \mathfrak{a} and $\langle \mathfrak{b}_\lambda^{i,j} : \lambda \in \mathfrak{a} \rangle$. But the latter belongs to N_{j+1} by $(*)_4(\gamma)$, so the former belongs to N_{j+1} and as $\langle \langle \mathfrak{b}_\lambda^{i,j,\epsilon} : \lambda \in \mathfrak{a} \rangle : \epsilon < |\mathfrak{a}|^+ \rangle$ is eventually constant, also each member of the sequence belongs to N_{j+1} . As also $\langle \langle f_\alpha^{a,\lambda} : \alpha < \lambda \rangle : \lambda \in \text{pcf}(\mathfrak{a}) \rangle$ belongs to N_{j+1} we clearly get that

$$\langle \langle \langle f_\alpha^{a,\lambda,\epsilon} : \epsilon < |\mathfrak{a}|^+ \rangle : \alpha < \lambda \rangle : \lambda \in \mathfrak{a} \rangle$$

belongs to N_{j+1} . Next we prove by induction on ϵ that, for $\lambda \in \mathfrak{a}$, we have:

$$\otimes_1 \quad \theta \in \mathfrak{b}_\lambda^{i,j,\epsilon} \text{ and } \lambda \in \mathfrak{a} \Rightarrow f_{\sup(N_\kappa \cap \lambda)}^{a,\lambda,\epsilon}(\theta) = \sup(N_\kappa \cap \theta).$$

For $\epsilon = 0$ this holds by $(*)_6$. For ϵ limit this holds by the induction hypothesis and the definition of $f_\alpha^{a,\lambda,\epsilon}$ (as union of earlier ones). For $\epsilon + 1$, we check $f_{\sup(N_\kappa \cap \lambda)}^{a,\lambda,\epsilon+1}(\theta)$ according to the case in its definition; for Case 1 use the induction hypothesis applied to $f_{\sup(N_\kappa \cap \lambda)}^{a,\lambda,\epsilon}$. For Case 2 (with μ), by the induction hypothesis applied to $f_{\sup(N_\kappa \cap \mu)}^{a,\mu,\epsilon}$.

Lastly, for Case 3 (with θ) we should note:

$$(i) \mathfrak{b}_\lambda^{i,j,\epsilon} \cap \mathfrak{b}_\theta[\mathfrak{a}] \notin J_{<\theta}[\mathfrak{a}].$$

[Why? By the case's assumption $\mathfrak{b}_\lambda^{i,j,\epsilon} \in (J_\theta[\mathfrak{a}])^+$ and $(*)_4(\alpha)$ above.]

$$(ii) f_{\sup(N_\kappa \cap \lambda)}^{\mathfrak{a},\lambda,\epsilon} \upharpoonright (\mathfrak{b}_\lambda^{i,j,\epsilon} \cap \mathfrak{b}_\theta^{i,j,\epsilon}) \subseteq f_{\sup(N_\kappa \cap \theta)}^{\mathfrak{a},\theta,\epsilon}.$$

[Why? By the induction hypothesis for ϵ , used concerning λ and θ .]

Hence (by the definition in case 3 and (i) + (ii)),

$$(iii) f_{\sup(N_\kappa \cap \lambda)}^{\mathfrak{a},\lambda,\epsilon+1}(\theta) \leq \sup(N_\kappa \cap \theta).$$

Now if $\gamma < \sup(N_\kappa \cap \theta)$ then for some $\gamma(1)$ we have $\gamma < \gamma(1) \in N_\kappa \cap \theta$, so letting $\mathfrak{b} =: \mathfrak{b}_\lambda^{i,j,\epsilon} \cap \mathfrak{b}_\theta[\mathfrak{a}] \cap \mathfrak{b}_\theta^{i,j,\epsilon}$, it belongs to $J_{\leq \theta}[\mathfrak{a}] \setminus J_{<\theta}[\mathfrak{a}]$ and we have

$$f_\gamma^{\mathfrak{a},\theta} \upharpoonright \mathfrak{b} <_{J_{<\theta}[\mathfrak{a}]} f_{\gamma(1)}^{\mathfrak{a},\theta} \upharpoonright \mathfrak{b} \leq f_{\sup(N_\kappa \cap \theta)}^{\mathfrak{a},\theta,\epsilon}$$

hence $f_{\sup(N_\kappa \cap \lambda)}^{\mathfrak{a},\lambda,\epsilon+1}(\theta) > \gamma$; as this holds for every $\gamma < \sup(N_\kappa \cap \theta)$ we have obtained

$$(iv) f_{\sup(N_\kappa \cap \lambda)}^{\mathfrak{a},\lambda,\epsilon+1}(\theta) \geq \sup(N_\kappa \cap \theta);$$

together we have finished proving the inductive step for $\epsilon+1$, hence we have proved \otimes_1 .

This is enough for proving $\mathfrak{b}_\lambda^{i,j,\epsilon} \in J_{\leq \lambda}[\mathfrak{a}]$.

Why? If it fails, as $\mathfrak{b}_\lambda^{i,j,\epsilon} \in N_{j+1}$ and $\langle f_\alpha^{\mathfrak{a},\lambda,\epsilon} : \alpha < \lambda \rangle$ belongs to N_{j+1} , there is $g \in \prod \mathfrak{b}_\lambda^{i,j,\epsilon}$ such that

$$(*) \quad \alpha < \lambda \Rightarrow f_\alpha^{\mathfrak{a},\lambda,\epsilon} \upharpoonright \mathfrak{b}^{i,j,\epsilon} < g \text{ mod } J_{\leq \lambda}[\mathfrak{a}].$$

Without loss of generality $g \in N_{j+1}$; by $(*)$, $f_{\sup(N_\kappa \cap \lambda)}^{\mathfrak{a},\lambda,\epsilon} < g \text{ mod } J_{\leq \lambda}[\mathfrak{a}]$. But $g < \langle \sup(N_\kappa \cap \theta) : \theta \in \mathfrak{b}_\lambda^{i,j,\epsilon} \rangle$. Together this contradicts \otimes_1 !

This ends the proof of 2.1. $\square_{2.1}$

If $|\text{pcf}(\mathfrak{a})| < \text{Min}(\mathfrak{a})$ then 2.1 is fine and helpful. But as we do not know this, we shall use the following substitute.

Claim 2.4. *Assume $|\mathfrak{a}| < \kappa = \text{cf}(\kappa) < \text{Min}(\mathfrak{a})$ and σ is an infinite ordinal satisfying $|\sigma|^+ < \kappa$. Let \bar{f} , $\bar{N} = \langle N_i : i < \kappa \rangle$, N_κ be as in the proof of 2.1. Then we can find $\bar{i} = \langle i_\alpha : \alpha \leq \sigma \rangle$, $\bar{\mathfrak{a}} = \langle \mathfrak{a}_\alpha : \alpha < \sigma \rangle$ and $\langle \langle \mathfrak{b}_\lambda^\beta[\bar{\mathfrak{a}}] : \lambda \in \mathfrak{a}_\beta \rangle : \beta < \sigma \rangle$ such that:*

- (a) \bar{i} is a strictly increasing continuous sequence of ordinals $< \kappa$,
- (b) for $\beta < \sigma$ we have $\langle i_\alpha : \alpha \leq \beta \rangle \in N_{i_{\beta+1}}$ hence $\langle N_{i_\alpha} : \alpha \leq \beta \rangle \in N_{i_{\beta+1}}$ and $\langle \mathfrak{b}_\lambda^\gamma[\bar{\mathfrak{a}}] : \lambda \in \mathfrak{a}_\gamma \text{ and } \gamma \leq \beta \rangle \in N_{i_{\beta+1}}$, we can get $\bar{i} \upharpoonright (\beta+1) \in N_{i_{\beta+1}}$ if κ successor of regular (we just need a suitable partial square)
- (c) $\mathfrak{a}_\beta = N_{i_\beta} \cap \text{pcf}(\mathfrak{a})$, so \mathfrak{a}_β is increasing continuous with β , $\mathfrak{a} \subseteq \mathfrak{a}_\beta \subseteq \text{pcf}(\mathfrak{a})$ and $|\mathfrak{a}_\beta| < \kappa$,
- (d) $\mathfrak{b}_\lambda^\beta[\bar{\mathfrak{a}}] \subseteq \mathfrak{a}_\beta$ (for $\lambda \in \mathfrak{a}_\beta$),
- (e) $J_{\leq \lambda}[\mathfrak{a}_\beta] = J_{<\lambda}[\mathfrak{a}_\beta] + \mathfrak{b}_\lambda^\beta[\bar{\mathfrak{a}}]$ (so $\lambda \in \mathfrak{b}_\lambda^\beta[\mathfrak{a}]$ and $\mathfrak{b}_\lambda^\beta[\bar{\mathfrak{a}}] \subseteq \lambda^+$),
- (f) if $\mu < \lambda$ are from \mathfrak{a}_β and $\mu \in \mathfrak{b}_\lambda^\beta[\bar{\mathfrak{a}}]$ then $\mathfrak{b}_\mu^\beta[\bar{\mathfrak{a}}] \subseteq \mathfrak{b}_\lambda^\beta[\bar{\mathfrak{a}}]$ (i.e., smoothness),
- (g) $\mathfrak{b}_\lambda^\beta[\bar{\mathfrak{a}}] = \mathfrak{a}_\beta \cap \text{pcf}(\mathfrak{b}_\lambda^\beta[\bar{\mathfrak{a}}])$ (i.e., closedness),

- (h) if $\mathfrak{c} \subseteq \mathfrak{a}_\beta, \beta < \sigma$ and $\mathfrak{c} \in N_{i_{\beta+1}}$ then for some finite $\mathfrak{d} \subseteq \mathfrak{a}_{\beta+1} \cap \text{pcf}(\mathfrak{c})$, we have $\mathfrak{c} \subseteq \bigcup_{\mu \in \mathfrak{d}} \mathfrak{b}_\mu^{\beta+1}[\bar{\mathfrak{a}}]$;

more generally (note that in (h)⁺ if $\theta = \aleph_0$ then we get (h)).

- (h)⁺ if $\mathfrak{c} \subseteq \mathfrak{a}_\beta, \beta < \sigma, \mathfrak{c} \in N_{i_{\beta+1}}, \theta = \text{cf}(\theta) \in N_{i_{\beta+1}}$, then for some $\mathfrak{d} \in N_{i_{\beta+1}}, \mathfrak{d} \subseteq \mathfrak{a}_{\beta+1} \cap \text{pcf}_{\theta\text{-complete}}(\mathfrak{c})$ we have $\mathfrak{c} \subseteq \bigcup_{\mu \in \mathfrak{d}} \mathfrak{b}_\mu^{\beta+1}[\bar{\mathfrak{a}}]$ and $|\mathfrak{d}| < \theta$,

- (i) $\mathfrak{b}_\lambda^\beta[\bar{\mathfrak{a}}]$ increases with β .

This will be proved below.

Claim 2.5. In 2.4 we can also have:

- (1) if we let $\mathfrak{b}_\lambda[\bar{\mathfrak{a}}] = \mathfrak{b}_\lambda^\sigma[\bar{\mathfrak{a}}] = \bigcup_{\beta < \sigma} \mathfrak{b}_\lambda^\beta[\bar{\mathfrak{a}}]$, $\mathfrak{a}_\sigma = \bigcup_{\beta < \sigma} \mathfrak{a}_\beta$ then also for $\beta = \sigma$ we have
 - (b) (use $N_{i_{\beta+1}}$), (c), (d), (f), (i)
- (2) If $\sigma = \text{cf}(\sigma) > |\mathfrak{a}|$ then for $\beta = \sigma$ also (e), (g)
- (3) If $\text{cf}(\sigma) > |\mathfrak{a}|, \mathfrak{c} \in N_{i_\sigma}, \mathfrak{c} \subseteq \mathfrak{a}_\sigma$ (hence $|\mathfrak{c}| < \text{Min}(\mathfrak{c})$ and $\mathfrak{c} \subseteq \mathfrak{a}_\sigma$), then for some finite $\mathfrak{d} \subseteq (\text{pcf}(\mathfrak{c})) \cap \mathfrak{a}_\sigma$ we have $\mathfrak{c} \subseteq \bigcup_{\mu \in \mathfrak{d}} \mathfrak{b}_\mu[\bar{\mathfrak{a}}]$. Similarly for θ -complete, $\theta < \text{cf}(\sigma)$ (i.e., we have clauses (h), (h)⁺ for $\beta = \sigma$).
- (4) We can have continuity in $\delta \leq \sigma$ when $\text{cf}(\delta) > |\mathfrak{a}|$, i.e., $\mathfrak{b}_\lambda^\delta[\bar{\mathfrak{a}}] = \bigcup_{\beta < \delta} \mathfrak{b}_\lambda^\beta[\bar{\mathfrak{a}}]$.

We shall prove 2.5 after proving 2.4.

Remark 2.6. 1) If we would like to use length κ , use \bar{N} as produced in [Sh:420, 2.6] so $\sigma = \kappa$.

2) Concerning 2.5, in 2.6(1) for a club E of $\sigma = \kappa$, we have $\alpha \in E \Rightarrow \mathfrak{b}_\lambda^\alpha[\bar{\mathfrak{a}}] = \mathfrak{b}_\lambda[\bar{\mathfrak{a}}] \cap \mathfrak{a}_\alpha$.

3) We can also use 2.4, 2.5 to give an alternative proof of part of the localization theorems similar to the one given in the Spring '89 lectures.

For example:

Claim 2.7. 1) If $|\mathfrak{a}| < \theta = \text{cf}(\theta) < \text{Min}(\mathfrak{a})$, for no sequence $\langle \lambda_i : i < \theta \rangle$ of members of $\text{pcf}(\mathfrak{a})$, do we have $\bigwedge_{\alpha < \theta} [\lambda_\alpha > \max \text{pcf}\{\lambda_i : i < \alpha\}]$.

2) If $|\mathfrak{a}| < \text{Min}(\mathfrak{a}), |\mathfrak{b}| < \text{Min}(\mathfrak{b}), \mathfrak{b} \subseteq \text{pcf}(\mathfrak{a})$ and $\lambda \in \text{pcf}(\mathfrak{a})$, then for some $\mathfrak{c} \subseteq \mathfrak{b}$ we have $|\mathfrak{c}| \leq |\mathfrak{a}|$ and $\lambda \in \text{pcf}(\mathfrak{c})$.

Proof. Relying on 2.4:

1) Without loss of generality $\text{Min}(\mathfrak{a}) > \theta^{+3}$, let $\kappa = \theta^{+2}$, let $\bar{N}, N_\kappa, \bar{\mathfrak{a}}, \mathfrak{b}$ (as a function), $\langle i_\alpha : \alpha \leq \sigma =: |\mathfrak{a}|^+ \rangle$ be as in 2.4 but we in addition assume that $\langle \lambda_i : i < \theta \rangle \in N_0$. So for $j < \theta$, $\mathfrak{c}_j =: \{\lambda_i : i < j\} \in N_0$ (so $\mathfrak{c}_j \subseteq \text{pcf}(\mathfrak{a}) \cap N_0 = \mathfrak{a}_0$) hence (by clause (h) of 2.4), for some finite $\mathfrak{d}_j \subseteq \mathfrak{a}_1 \cap \text{pcf}(\mathfrak{c}_j) = N_{i_1} \cap \text{pcf}(\mathfrak{a}) \cap \text{pcf}(\mathfrak{c}_j)$ we have $\mathfrak{c}_j \subseteq \bigcup_{\lambda \in \mathfrak{d}_j} \mathfrak{b}_\lambda^1[\bar{\mathfrak{a}}]$. Assume $j(1) < j(2) < \theta$. Now if $\mu \in \mathfrak{a} \cap \bigcup_{\lambda \in \mathfrak{d}_{j(1)}} \mathfrak{b}_\lambda^1[\bar{\mathfrak{a}}]$ then

for some $\mu_0 \in \mathfrak{d}_{j(1)}$ we have $\mu \in \mathfrak{b}_{\mu_0}^1[\bar{\mathfrak{a}}]$; now $\mu_0 \in \mathfrak{d}_{j(1)} \subseteq \text{pcf}(\mathfrak{c}_{j(1)}) \subseteq \text{pcf}(\mathfrak{c}_{j(2)}) \subseteq \text{pcf}(\bigcup_{\lambda \in \mathfrak{d}_{j(2)}} \mathfrak{b}_\lambda^1[\bar{\mathfrak{a}}]) = \bigcup_{\lambda \in \mathfrak{d}_{j(2)}} (\text{pcf}(\mathfrak{b}_\lambda^1[\bar{\mathfrak{a}}]))$ hence (by clause (g) of 2.4 as $\mu_0 \in \mathfrak{d}_{j(0)} \subseteq N_1$)

for some $\mu_1 \in \mathfrak{d}_{j(2)}, \mu_0 \in \mathfrak{b}_{\mu_1}^1[\bar{\mathfrak{a}}]$. So by clause (f) of 2.4 we have $\mathfrak{b}_{\mu_0}^1[\bar{\mathfrak{a}}] \subseteq \mathfrak{b}_{\mu_1}^1[\bar{\mathfrak{a}}]$ hence remembering $\mu \in \mathfrak{b}_{\mu_0}^1[\bar{\mathfrak{a}}]$, we have $\mu \in \mathfrak{b}_{\mu_1}^1[\bar{\mathfrak{a}}]$. Remembering μ was any member of

$\mathfrak{a} \cap \bigcup_{\lambda \in \mathfrak{d}_{j(1)}} \mathfrak{b}_\lambda^1[\bar{\mathfrak{a}}]$, we have $\mathfrak{a} \cap \bigcup_{\lambda \in \mathfrak{d}_{j(1)}} \mathfrak{b}_\lambda^1[\bar{\mathfrak{a}}] \subseteq \mathfrak{a} \cap \bigcup_{\lambda \in \mathfrak{d}_{j(2)}} \mathfrak{b}_\lambda^1[\bar{\mathfrak{a}}]$ (holds also without “ $\mathfrak{a} \cap$ ” but not used). So $\langle \mathfrak{a} \cap \bigcup_{\lambda \in \mathfrak{d}_j} \mathfrak{b}_\lambda^1[\bar{\mathfrak{a}}] : j < \theta \rangle$ is a \subseteq -increasing sequence of subsets of \mathfrak{a} , but $\text{cf}(\theta) > |\mathfrak{a}|$, so the sequence is eventually constant, say for $j \geq j(*)$. But

$$\begin{aligned} \max \text{pcf}(\mathfrak{a} \cap \bigcup_{\lambda \in \mathfrak{d}_j} \mathfrak{b}_\lambda^1[\bar{\mathfrak{a}}]) &\leq \max \text{pcf}(\bigcup_{\lambda \in \mathfrak{d}_j} \mathfrak{b}_\lambda^1[\bar{\mathfrak{a}}]) \\ &= \max_{\lambda \in \mathfrak{d}_j} (\max \text{pcf}(\mathfrak{b}_\lambda^1[\bar{\mathfrak{a}}])) \\ &= \max_{\lambda \in \mathfrak{d}_j} \lambda \leq \max \text{pcf}\{\lambda_i : i < j\} < \lambda_j \\ &= \max \text{pcf}(\mathfrak{a} \cap \bigcup_{\lambda \in \mathfrak{d}_{j+1}} \mathfrak{b}_\lambda^1[\bar{\mathfrak{a}}]) \end{aligned}$$

(last equality as $\mathfrak{b}_{\lambda_j}[\bar{\mathfrak{a}}] \subseteq \mathfrak{b}_\lambda^1[\bar{\mathfrak{a}}] \pmod{J_{<\lambda}[\mathfrak{a}_1]}$). Contradiction.

2) (Like [Sh:371, §3]): If this fails choose a counterexample \mathfrak{b} with $|\mathfrak{b}|$ minimal, and among those with $\max \text{pcf}(\mathfrak{b})$ minimal and among those with $\bigcup \{\mu^+ : \mu \in \lambda \cap \text{pcf}(\mathfrak{b})\}$ minimal. So by the pcf theorem

- (*)₁ $\text{pcf}(\mathfrak{b}) \cap \lambda$ has no last member
- (*)₂ $\mu = \sup[\lambda \cap \text{pcf}(\mathfrak{b})]$ is not in $\text{pcf}(\mathfrak{b})$ or $\mu = \lambda$.
- (*)₃ $\max \text{pcf}(\mathfrak{b}) = \lambda$.

Try to choose by induction on $i < |\mathfrak{a}|^+$, $\lambda_i \in \lambda \cap \text{pcf}(\mathfrak{b})$, $\lambda_i > \max \text{pcf}\{\lambda_j : j < i\}$. Clearly by part (1), we will be stuck at some i . Now $\text{pcf}\{\lambda_j : j < i\}$ has a last member and is included in $\text{pcf}(\mathfrak{b})$, hence by (*)₃ and being stuck at necessarily $\text{pcf}(\{\lambda_j : j < i\}) \not\subseteq \lambda$ but it is $\subseteq \text{pcf}(\mathfrak{b}) \subseteq \lambda^+$, so $\lambda = \max \text{pcf}\{\lambda_j : j < i\}$. For each j , by the choice of “minimal counterexample” for some $\mathfrak{b}_j \subseteq \mathfrak{b}$, we have $|\mathfrak{b}_j| \leq |\mathfrak{a}|$, $\lambda_j \in \text{pcf}(\mathfrak{b}_j)$. So $\lambda \in \text{pcf}\{\lambda_j : j < i\} \subseteq \text{pcf}(\bigcup_{j < i} \mathfrak{b}_j)$ but $\bigcup_{j < i} \text{frb}_j$ is a subset of \mathfrak{b} of cardinality $\leq |i| \times |\mathfrak{a}| = |\mathfrak{a}|$, so we are done. $\square_{2.6}$

Proof. Without loss of generality $\sigma = \omega\sigma$ (as we can use $\omega^\omega\sigma$ so $|\omega^\omega\sigma| = |\sigma|$). Let $\bar{f}^{\mathfrak{a}} = \langle \bar{f}^{\mathfrak{a},\lambda} : \langle \langle f_\alpha^{\mathfrak{a},\lambda} : \alpha < \lambda \rangle : \lambda \in \text{pcf}(\mathfrak{a}) \rangle \rangle$ and $\langle N_i : i \leq \kappa \rangle$ be chosen as in the proof of 2.1 and without loss of generality $\bar{f}^{\mathfrak{a}}$ belongs to N_0 . For $\zeta < \kappa$ we define $\mathfrak{a}^\zeta =: N_\zeta \cap \text{pcf}(\mathfrak{a})$; we also define ${}^\zeta\bar{f}$ as $\langle \langle f_\alpha^{\mathfrak{a}^\zeta,\lambda} : \alpha < \lambda \rangle : \lambda \in \text{pcf}(\mathfrak{a}) \rangle \rangle$ where $f_\alpha^{\mathfrak{a}^\zeta,\lambda} \in \prod \mathfrak{a}^\zeta$ is defined as follows:

- (a) if $\theta \in \mathfrak{a}$, $f_\alpha^{\mathfrak{a}^\zeta,\lambda}(\theta) = f_\alpha^{\mathfrak{a},\lambda}(\theta)$,
- (b) if $\theta \in \mathfrak{a}^\zeta \setminus \mathfrak{a}$ and $\text{cf}(\alpha) \notin (|\mathfrak{a}^\zeta|, \text{Min}(\mathfrak{a}))$, then

$$f_\alpha^{\mathfrak{a}^\zeta,\lambda}(\theta) = \text{Min}\{\gamma < \theta : f_\alpha^{\mathfrak{a},\lambda} \upharpoonright \mathfrak{b}_\theta[\mathfrak{a}] \leq_{J_{<\theta}[\mathfrak{b}_\theta[\mathfrak{a}]]} f_\gamma^{\mathfrak{a},\theta} \upharpoonright \mathfrak{b}_\theta[\mathfrak{a}]\},$$

- (c) if $\theta \in \mathfrak{a}^\zeta \setminus \mathfrak{a}$ and $\text{cf}(\alpha) \in (|\mathfrak{a}^\zeta|, \text{Min}(\mathfrak{a}))$, define $f_\alpha^{\mathfrak{a}^\zeta,\lambda}(\theta)$ so as to satisfy (*)₁ in the proof of 2.1.

Now ${}^\zeta\bar{f}$ is legitimate except that we have only

$$\beta < \gamma < \lambda \in \text{pcf}(\mathfrak{a}) \Rightarrow f_\beta^{\mathfrak{a}^\zeta,\lambda} \leq f_\gamma^{\mathfrak{a}^\zeta,\lambda} \pmod{J_{<\lambda}[\mathfrak{a}^\zeta]}$$

(instead of strict inequality) however we still have $\bigwedge_{\beta < \lambda} \bigvee_{\gamma < \lambda} [f_\beta^{\mathfrak{a}^\zeta, \lambda} < f_\gamma^{\mathfrak{a}^\zeta, \lambda} \text{ mod } J_{< \lambda}[\mathfrak{a}^\zeta]]$,

but this suffices. (The first statement is actually proved in [Sh:371, 3.2A], the second in [Sh:371, 3.2B]; by it also ${}^\zeta \bar{f}$ is cofinal in the required sense.)

For every $\zeta < \kappa$ we can apply the proof of 2.1 with $(N_\zeta \cap \text{pcf}(\mathfrak{a}))$, ${}^\zeta \bar{f}$ and $\langle N_{\zeta+1+i} : i < \kappa \rangle$ here standing for \mathfrak{a} , \bar{f} , \bar{N} there. In the proof of 2.1 get a club E^ζ of κ (corresponding to E there and without loss of generality $\zeta + \text{Min}(E^\zeta) = \text{Min}(E^\zeta)$) so any $i < j$ from E^ζ are O.K.). Now we can define for $\zeta < \kappa$ and $i < j$ from E^ζ , ${}^\zeta \mathfrak{b}_\lambda^{i,j}$ and $\langle {}^\zeta \mathfrak{b}_\lambda^{i,j,\epsilon} : \epsilon < |\mathfrak{a}^\zeta|^+ \rangle$, $\langle \epsilon^\zeta(i, j, \lambda) : \lambda \in \mathfrak{a}^\zeta \rangle$, $\epsilon^\zeta(i, j)$, as well as in the proof of 2.1.

Let:

$$E = \{i < \kappa : i \text{ is a limit ordinal } (\forall j < i)(j + j < i \text{ and } j \times j < i) \\ \text{and } \bigwedge_{j < i} i \in E^j\}.$$

So by [Sh:420, §1] we can find $\bar{C} = \langle C_\delta : \delta \in S \rangle$, $S \subseteq \{\delta < \kappa : \text{cf}(\delta) = \text{cf}(\sigma)\}$ stationary, C_δ a club of δ , $\text{otp}(C_\delta) = \sigma$ such that:

- (1) for each $\alpha < \lambda$, $\{C_\delta \cap \alpha : \alpha \in \text{nacc}(C_\delta)\}$ has cardinality $< \kappa$. If κ is successor of regular, then we can get $[\gamma \in C_\alpha \cap C_\beta \Rightarrow C_\alpha \cap \gamma = C_\beta \cap \gamma]$ and
- (2) for every club E' of κ for stationarily many $\delta \in S$, $C_\delta \subseteq E'$.

Without loss of generality $\bar{C} \in N_0$. For some δ^* , $C_{\delta^*} \subseteq E$, and let $\{j_\zeta : \zeta \leq \omega^2 \sigma\}$ enumerate $C_{\delta^*} \cup \{\delta^*\}$. So $\langle j_\zeta : \zeta \leq \omega^2 \sigma \rangle$ is a strictly increasing continuous sequence of ordinals from $E \subseteq \kappa$ such that $\langle j_\epsilon : \epsilon \leq \zeta \rangle \in N_{j_{\zeta+1}}$ and if, e.g., κ is a successor of regulars then $\langle j_\epsilon : \epsilon \leq \zeta \rangle \in N_{j_{\zeta+1}}$. Let $j(\zeta) = j_\zeta$ and for $\ell \in \{0, 2\}$ let $i_\ell(\zeta) = i_\zeta^\ell =: j_{\omega^\ell(1+\zeta)}$, $\mathfrak{a}_\zeta = N_{i_\zeta}^\ell \cap \text{pcf}(\mathfrak{a})$, and $\bar{\mathfrak{a}}^\ell =: \langle \mathfrak{a}_\zeta^\ell : \zeta < \sigma \rangle$, ${}^\ell \mathfrak{b}_\lambda^\zeta[\bar{\mathfrak{a}}] =: {}_{i_\ell(\zeta)} \mathfrak{b}_\lambda^{j(\omega^\ell \zeta+1), j(\omega^\ell \zeta+2), \epsilon^\zeta(j(\omega^\ell \zeta+1), j(\omega^\ell \zeta+2))}$. Recall that $\sigma = \omega \sigma$ so $\sigma = \omega^2 \sigma$; if the value of ℓ does not matter we omit it. Most of the requirements follow immediately by the proof of 2.1, as

- ⊗ for each $\zeta < \sigma$, we have \mathfrak{b}_ζ , $\langle \mathfrak{b}_\lambda^\zeta[\bar{\mathfrak{a}}] : \lambda \in \mathfrak{a}_\zeta \rangle$ are as in the proof (hence conclusion of 2.1) and belongs to $N_{i_{\beta+3}} \subseteq N_{i_{\beta+1}}$.

We are left (for proving 2.4) with proving clauses (h)⁺ and (i) (remember that (h) is a special case of (h)⁺ choosing $\theta = \aleph_0$).

For proving clause (i) note that for $\zeta < \xi < \kappa$, $f_\alpha^{\mathfrak{a}^\zeta, \lambda} \subseteq f_\alpha^{\mathfrak{a}^\xi, \lambda}$ hence ${}^\zeta \mathfrak{b}_\lambda^{i,j} \subseteq {}^\xi \mathfrak{b}_\lambda^{i,j}$. Now we can prove by induction on ϵ that ${}^\zeta \mathfrak{b}_\lambda^{i,j,\epsilon} \subseteq {}^\xi \mathfrak{b}_\lambda^{i,j,\epsilon}$ for every $\lambda \in \mathfrak{a}_\zeta$ (check the definition in $(*)_7$ in the proof of 2.1) and the conclusion follows.

Instead of proving (h)⁺ we prove an apparently weaker version (h)' below, but having (h)' for the case $\ell = 0$ gives (h)⁺ for $\ell = 2$ so this is enough [[then note that $\bar{i}' = \langle i_{\omega^2 \zeta} : \zeta < \sigma \rangle$, $\bar{\mathfrak{a}}' = \langle \mathfrak{a}_{\omega^2 \zeta} : \zeta < \sigma \rangle$, $\langle N_{i_{\omega^2 \zeta}} : \zeta < \sigma \rangle$, $\langle \mathfrak{b}_\lambda^{\omega^2 \zeta}[\bar{\mathfrak{a}}'] : \zeta < \sigma, \lambda \in \mathfrak{a}'_\zeta = \mathfrak{a}_{\omega^2 \zeta} \rangle$ will exemplify the conclusion]] where:

- (h)' if $\mathfrak{c} \subseteq \mathfrak{a}_\beta$, $\beta < \sigma$, $\mathfrak{c} \in N_{i_{\beta+1}}$, $\theta = \text{cf}(\theta) \in N_{i_{\beta+1}}$ then for some frd $\in N_{i_{\beta+\omega+1}+1}$ satisfying $\mathfrak{d} \subseteq \mathfrak{a}_{\beta+\omega} \cap \text{pcf}_{\theta\text{-complete}}(\mathfrak{c})$ we have $\mathfrak{c} \subseteq \bigcup_{\mu \in \mathfrak{d}} \mathfrak{b}_\mu^{\beta+\omega}[\bar{\mathfrak{a}}]$ and $|\mathfrak{d}| < \theta$.

□

Proof. Proof of (h)'

So let $\theta, \beta, \mathfrak{c}$ be given; let $\langle \mathfrak{b}_\mu[\bar{\mathfrak{a}}] : \mu \in \text{pcf}(\mathfrak{c}) \rangle (\in N_{i_{\beta+1}})$ be a generating sequence. We define by induction on $n < \omega$, $A_n, \langle \langle \mathfrak{c}_\eta, \lambda_\eta \rangle : \eta \in A_n \rangle$ such that:

- (a) $A_0 = \{\langle \rangle\}, \mathfrak{c}_{\langle \rangle} = \mathfrak{c}, \lambda_{\langle \rangle} = \max \text{pcf}(\mathfrak{c})$,
- (b) $A_n \subseteq {}^n\theta, |A_n| < \theta$,
- (c) if $\eta \in A_{n+1}$ then $\eta \upharpoonright n \in A_n, \mathfrak{c}_\eta \subseteq \mathfrak{c}_{\eta \upharpoonright n}, \lambda_\eta < \lambda_{\eta \upharpoonright n}$ and $\lambda_\eta = \max \text{pcf}(\mathfrak{c}_\eta)$,
- (d) $A_n, \langle \langle \mathfrak{c}_\eta, \lambda_\eta \rangle : \eta \in A_n \rangle$ belongs to $N_{i_{\beta+1+n}}$ hence $\lambda_\eta \in N_{i_{\beta+1+n}}$,
- (e) if $\eta \in A_n$ and $\lambda_\eta \in \text{pcf}_{\theta\text{-complete}}(\mathfrak{c}_\eta)$ and $\mathfrak{c}_\eta \not\subseteq \mathfrak{b}_{\lambda_\eta}^{\beta+1+n}[\bar{\mathfrak{a}}]$ then $(\forall \nu)[\nu \in A_{n+1} \text{ and } \eta \subseteq \nu \Leftrightarrow \nu = \eta^\wedge \langle 0 \rangle]$ and $\mathfrak{c}_{\eta^\wedge \langle 0 \rangle} = \mathfrak{c}_\eta \setminus \mathfrak{b}_{\lambda_\eta}^{\beta+1+n}[\bar{\mathfrak{a}}]$ (so $\lambda_{\eta^\wedge \langle 0 \rangle} = \max \text{pcf}(\mathfrak{c}_{\eta^\wedge \langle 0 \rangle}) < \lambda_\eta = \max \text{pcf}(\mathfrak{c}_\eta)$,
- (f) if $\eta \in A_n$ and $\lambda_\eta \notin \text{pcf}_{\theta\text{-complete}}(\mathfrak{c}_\eta)$ then

$$\mathfrak{c}_\eta = \bigcup \{ \mathfrak{b}_{\lambda_{\eta^\wedge \langle i \rangle}}[c] : i < i_n < \theta, \eta^\wedge \langle i \rangle \in A_{n+1} \},$$

and if $\nu = \eta^\wedge \langle i \rangle \in A_{n+1}$ then $\mathfrak{c}_\nu = \mathfrak{b}_{\lambda_\nu}[c]$,

- (g) if $\eta \in A_n$, and $\lambda_\eta \in \text{pcf}_{\theta\text{-complete}}(\mathfrak{c}_\eta)$ but $\mathfrak{c}_\eta \subseteq \mathfrak{b}_{\lambda_\eta}^{\beta+1-n}[\bar{\mathfrak{a}}]$, then $\neg(\exists \nu)[\eta \triangleleft \nu \in A_{n+1}]$.

There is no problem to carry the definition (we use 2.8(1), the point is that $\mathfrak{c} \in N_{i_{\beta+1+n}}$ implies $\langle \mathfrak{b}_\lambda(\mathfrak{c}) : \lambda \in \text{pcf}_\theta[\mathfrak{c}] \rangle \in N_{i_{\beta+1+n}}$ and as there is \mathfrak{d} as in 2.8(1), there is one in $N_{i_{\beta+1+n+1}}$ so $\mathfrak{d} \subseteq \mathfrak{a}_{\beta+1+n+1}$).

Now let

$$\mathfrak{d}_n =: \{ \lambda_\eta : \eta \in A_n \text{ and } \lambda_\eta \in \text{pcf}_{\theta\text{-complete}}(\mathfrak{c}_\eta) \}$$

and $\mathfrak{d} =: \bigcup_{n < \omega} \mathfrak{d}_n$; we shall show that it is as required.

The main point is $\mathfrak{c} \subseteq \bigcup_{\lambda \in \mathfrak{d}} \mathfrak{b}_\lambda^{\beta+\omega}[\bar{\mathfrak{a}}]$; note that

$$[\lambda_\eta \in \mathfrak{d}, \eta \in A_n \Rightarrow \mathfrak{b}_{\lambda_\eta}^{\beta+1+n}[\bar{\mathfrak{a}}] \subseteq \mathfrak{b}_{\lambda_\eta}^{\beta+\omega}[\bar{\mathfrak{a}}]]$$

hence it suffices to show $\mathfrak{c} \subseteq \bigcup_{n < \omega} \bigcup_{\lambda \in \mathfrak{d}_n} \mathfrak{b}_\lambda^{\beta+1+n}[\bar{\mathfrak{a}}]$, so assume $\theta \in \mathfrak{c} \setminus \bigcup_{n < \omega} \bigcup_{\lambda \in \mathfrak{d}_n} \mathfrak{b}_\lambda^{\beta+1+n}[\bar{\mathfrak{a}}]$,

and we choose by induction on n , $\eta_n \in A_n$ such that $\eta_0 = \langle \rangle$, $\eta_{n+1} \upharpoonright n = \eta_n$ and $\theta \in \mathfrak{c}_{\eta_n}$; by clauses (e) + (f) above this is possible and $\langle \max \text{pcf}(\mathfrak{c}_{\eta_n}) : n < \omega \rangle$ is (strictly) decreasing, contradiction.

The minor point is $|\mathfrak{d}| < \theta$; if $\theta > \aleph_0$ note that $\bigwedge_n |A_n| < \theta$ and $\theta = \text{cf}(\theta)$ clearly $|\mathfrak{d}| \leq |\bigcup_n A_n| < \theta + \aleph_1 = \theta$.

If $\theta = \aleph_0$ (i.e. clause (h)) we should show that $\bigcup_n A_n$ finite; the proof is as above noting that the clause (f) is vacuous now. So $n < \omega \Rightarrow |A_n| = 1$ and for some $n \bigvee_n A_n = \emptyset$, so $\bigcup_n A_n$ is finite. Another minor point is $\mathfrak{d} \in N_{i_{\beta+\omega+1}}$; this holds as the construction is unique from $\mathfrak{c}, \langle \mathfrak{b}_\mu[c] : \mu \in \text{pcf}(\mathfrak{c}) \rangle, \langle N_j : j < i_{\beta+\omega} \rangle, \langle i_j : j \leq \beta + \omega \rangle, \langle \langle \mathfrak{a}_{i(\zeta)}, \langle \mathfrak{b}_\lambda^\zeta[\bar{\mathfrak{a}}] : \lambda \in \mathfrak{a}_{i(\zeta)} \rangle \rangle : \zeta \leq \beta + \omega \rangle$; no “outside” information is used so $\langle \langle A_n, \langle \langle \mathfrak{c}_\eta, \lambda_\eta \rangle : \eta \in A_n \rangle \rangle : n < \omega \rangle \in N_{i_{\beta+\omega+1}}$, so (using a choice function) really $\mathfrak{d} \in N_{i_{\beta+\omega+1}}$. $\square_{2.4}$

Proof. Let $\mathfrak{b}_\lambda[\bar{\mathfrak{a}}] = \mathfrak{b}_\lambda^\sigma = \bigcup_{\beta < \sigma} \mathfrak{b}_\lambda^\beta[\bar{\mathfrak{a}}]$ and $\mathfrak{a}_\sigma = \bigcup_{\zeta < \sigma} \mathfrak{a}_\zeta$. Part (1) is straightforward.

For part (2), for clause (g), for $\beta = \sigma$, the inclusion “ \subseteq ” is straightforward; so assume $\mu \in \mathfrak{a}_\beta \cap \text{pcf}(\mathfrak{b}_\lambda^\beta[\bar{\mathfrak{a}}])$. Then by 2.4(c) for some $\beta_0 < \beta$, we have $\mu \in \mathfrak{a}_{\beta_0}$, and by 2.7 (which depends on 2.4 only) for some $\beta_1 < \beta$, $\mu \in \text{pcf}(\mathfrak{b}_\lambda^{\beta_1}[\bar{\mathfrak{a}}])$; by monotonicity without loss of generality $\beta_0 = \beta_1$, by clause (g) of 2.4 applied to β_0 , $\mu \in \mathfrak{b}_\lambda^{\beta_0}[\bar{\mathfrak{a}}]$. Hence by clause (i) of 2.4, $\mu \in \mathfrak{b}_\lambda^\beta[\bar{\mathfrak{a}}]$, thus proving the other inclusion.

The proof of clause (e) (for 2.5(2)) is similar, and also 2.5(3). For ?? for $\delta < \sigma$, $\text{cf}(\delta) > |\mathfrak{a}|$ redefine $\mathfrak{b}_\lambda^\delta[\bar{\mathfrak{a}}]$ as $\bigcup_{\beta < \delta} \mathfrak{b}_\lambda^{\beta+1}[\bar{\mathfrak{a}}]$. $\square_{2.5}$

Claim 2.8. *Let θ be regular.*

0) If $\alpha < \theta$, $\text{pcf}_{\theta\text{-complete}}(\bigcup_{i < \alpha} \mathfrak{a}_i) = \bigcup_{i < \alpha} \text{pcf}_{\theta\text{-complete}}(\mathfrak{a}_i)$.

1) If $\langle \mathfrak{b}_\theta[\mathfrak{a}] : \partial \in \text{pcf}(\mathfrak{a}) \rangle$ is a generating sequence for \mathfrak{a} , $\mathfrak{c} \subseteq \mathfrak{a}$, then for some $\mathfrak{d} \subseteq \text{pcf}_{\theta\text{-complete}}(\mathfrak{c})$ we have: $|\mathfrak{d}| < \theta$ and $\mathfrak{c} \subseteq \bigcup_{\theta \in \mathfrak{a}} \mathfrak{b}_\theta[\mathfrak{a}]$.

2) If $|\mathfrak{a} \cup \mathfrak{c}| < \text{Min}(\mathfrak{a})$, $\mathfrak{c} \subseteq \text{pcf}_{\theta\text{-complete}}(\mathfrak{a})$, $\lambda \in \text{pcf}_{\theta\text{-complete}}(\mathfrak{c})$ then $\lambda \in \text{pcf}_{\theta\text{-complete}}(\mathfrak{a})$.

3) In (2) we can weaken $|\mathfrak{a} \cup \mathfrak{c}| < \text{Min}(\mathfrak{a})$ to $|\mathfrak{a}| < \text{Min}(\mathfrak{a})$, $|\mathfrak{c}| < \text{Min}(\mathfrak{c})$.

Proof. (0) and (1): Left to the reader.

2) See [Sh:345b, 1.10–1.12].

3) Similarly. $\square_{2.8}$

Claim 2.9. 1) Let θ be regular $\leq |\mathfrak{a}|$. We cannot find $\lambda_\alpha \in \text{pcf}_{\theta\text{-complete}}(\mathfrak{a})$ for $\alpha < |\mathfrak{a}|^+$ such that $\lambda_i > \sup \text{pcf}_{\theta\text{-complete}}(\{\lambda_j : j < i\})$.

2) Assume $\theta \leq |\mathfrak{a}|$, $\mathfrak{c} \subseteq \text{pcf}_{\theta\text{-complete}}(\mathfrak{a})$ (and $|\mathfrak{c}| < \text{Min}(\mathfrak{c})$; of course $|\mathfrak{a}| < \text{Min}(\mathfrak{a})$). If $\lambda \in \text{pcf}_{\theta\text{-complete}}(\mathfrak{c})$ then for some $\mathfrak{d} \subseteq \mathfrak{c}$ we have $|\mathfrak{d}| \leq |\mathfrak{a}|$ and $\lambda \in \text{pcf}_{\theta\text{-complete}}(\mathfrak{d})$.

Proof. 1) If $\theta = \aleph_0$ we already know it (see 2.7), so assume $\theta > \aleph_0$. We use 2.4 with $\{\theta, \langle \lambda_i : i < |\mathfrak{a}|^+ \rangle\} \in N_0$, $\sigma = |\mathfrak{a}|^+$, $\kappa = |\mathfrak{a}|^{+3}$ where, without loss of generality, $\kappa < \text{Min}(\mathfrak{a})$. For each $\alpha < |\mathfrak{a}|^+$ by (h)⁺ of 2.4 there is $\mathfrak{a}_\alpha \in N_{i_1}$, $\mathfrak{d}_\alpha \subseteq \text{pcf}_{\theta\text{-complete}}(\{\lambda_i : i < \alpha\})$, $|\mathfrak{d}_\alpha| < \theta$ such that $\{\lambda_i : i < \alpha\} \subseteq \bigcup_{\theta \in \mathfrak{d}_\alpha} \mathfrak{b}_\theta^1[\bar{\mathfrak{a}}]$; hence by clause (g) of 2.4

and part (0) Claim 2.8 we have $\mathfrak{a}_1 \cap \text{pcf}_{\theta\text{-complete}}(\{\lambda_i : i < \alpha\}) \subseteq \bigcup_{\theta \in \mathfrak{d}_\alpha} \mathfrak{b}_\theta^1[\bar{\mathfrak{a}}]$. So

for $\alpha < \beta < |\mathfrak{a}|^+$, $\mathfrak{d}_\alpha \subseteq \mathfrak{a}_1 \cap \text{pcf}_{\theta\text{-complete}}\{\lambda_i : i < \alpha\} \subseteq \mathfrak{a}_1 \cap \text{pcf}_{\theta\text{-complete}}\{\lambda_i : i < \beta\} \subseteq \bigcup_{\theta \in \mathfrak{d}_\beta} \mathfrak{b}_\theta^1[\bar{\mathfrak{a}}]$. As the sequence is smooth (i.e., clause (f) of 2.4) clearly

$$\alpha < \beta \Rightarrow \bigcup_{\mu \in \mathfrak{d}_\alpha} \mathfrak{b}_\mu^1[\bar{\mathfrak{a}}] \subseteq \bigcup_{\mu \in \mathfrak{d}_\beta} \mathfrak{b}_\mu^1[\bar{\mathfrak{a}}].$$

So $\langle \bigcup_{\mu \in \mathfrak{d}_\alpha} \mathfrak{b}_\mu^1[\bar{\mathfrak{a}}] \cap \mathfrak{a} : \alpha < |\mathfrak{a}|^+ \rangle$ is a non-decreasing sequence of subsets of \mathfrak{a} of length $|\mathfrak{a}|^+$, hence for some $\alpha(*) < |\mathfrak{a}|^+$ we have:

$$(*)_1 \quad \alpha(*) \leq \alpha < |\mathfrak{a}|^+ \Rightarrow \bigcup_{\mu \in \mathfrak{d}_\alpha} \mathfrak{b}_\mu^1[\bar{\mathfrak{a}}] \cap \mathfrak{a} = \bigcup_{\mu \in \mathfrak{d}_{\alpha(*)}} \mathfrak{b}_\mu^1[\bar{\mathfrak{a}}] \cap \mathfrak{a}.$$

If $\tau \in \mathfrak{a}_1 \cap \text{pcf}_{\theta\text{-complete}}(\{\lambda_i : i < \alpha\})$ then $\tau \in \text{pcf}_{\theta\text{-complete}}(\mathfrak{a})$ (by parts (2),(3) of Claim 2.8), and $\tau \in \mathfrak{b}_{\mu_\tau}^1[\bar{\mathfrak{a}}]$ for some $\mu_\tau \in \mathfrak{d}_\alpha$ so $\mathfrak{b}_\tau^1[\bar{\mathfrak{a}}] \subseteq \mathfrak{b}_{\mu_\tau}^1[\bar{\mathfrak{a}}]$, also $\tau \in \text{pcf}_{\theta\text{-complete}}(\mathfrak{b}_\tau^1[\bar{\mathfrak{a}}] \cap \mathfrak{a})$ (by clause (e) of 2.4), hence

$$\begin{aligned} \tau \in \text{pcf}_{\theta\text{-complete}}(\mathfrak{b}_\tau^1[\bar{\mathfrak{a}}] \cap \mathfrak{a}) &\subseteq \text{pcf}_{\theta\text{-complete}}(\mathfrak{b}_{\mu_\tau}^1[\bar{\mathfrak{a}}] \cap \mathfrak{a}) \\ &\subseteq \text{pcf}_{\theta\text{-complete}}(\bigcup_{\mu \in \mathfrak{d}_\alpha} \mathfrak{b}_\mu^1[\bar{\mathfrak{a}}] \cap \mathfrak{a}). \end{aligned}$$

So $\mathfrak{a}_1 \cap \text{pcf}_{\theta\text{-complete}}(\{\lambda_i : i < \alpha\}) \subseteq \text{pcf}_{\theta\text{-complete}}(\bigcup_{\mu \in \mathfrak{d}_\alpha} \mathfrak{b}_\mu^1[\bar{\mathfrak{a}}] \cap \mathfrak{a})$. But for each $\alpha < |\mathfrak{a}|^+$ we have $\lambda_\alpha > \sup \text{pcf}_{\theta\text{-complete}}(\{\lambda_i : i < \alpha\})$, whereas $\mathfrak{d}_\alpha \subseteq \text{pcf}_{\sigma\text{-complete}}\{\lambda_i : i < \alpha\}$, hence $\lambda_\alpha > \sup \mathfrak{d}_\alpha$ hence

$$(*)_2 \quad \lambda_\alpha > \sup_{\mu \in \mathfrak{d}_\alpha} \max \text{pcf}(\mathfrak{b}_\mu^1[\bar{\mathfrak{a}}]) \geq \sup \text{pcf}_{\theta\text{-complete}}(\bigcup_{\mu \in \mathfrak{d}_\alpha} \mathfrak{b}_\mu^1[\bar{\mathfrak{a}}] \cap \mathfrak{a}).$$

On the other hand,

$$(*)_3 \quad \lambda_\alpha \in \text{pcf}_{\theta\text{-complete}}\{\lambda_i : i < \alpha + 1\} \subseteq \text{pcf}_{\theta\text{-complete}}(\bigcup_{\mu \in \mathfrak{d}_{\alpha+1}} \mathfrak{b}_\mu^1[\bar{\mathfrak{a}}] \cap \mathfrak{a}).$$

For $\alpha = \alpha(*)$ we get contradiction by $(*)_1 + (*)_2 + (*)_3$.

2) Assume $\mathfrak{a}, \mathfrak{c}, \lambda$ form a counterexample with λ minimal. Without loss of generality $|\mathfrak{a}|^{+3} < \text{Min}(\mathfrak{a})$ and $\lambda = \max \text{pcf}(\mathfrak{a})$ and $\lambda = \max \text{pcf}(\mathfrak{c})$ (just let $\mathfrak{a}' =: \mathfrak{b}_\lambda[\mathfrak{a}]$, $\mathfrak{c}' =: \mathfrak{c} \cap \text{pcf}_\theta[\mathfrak{a}']$; if $\lambda \notin \text{pcf}_{\theta\text{-complete}}(\mathfrak{c}')$ then necessarily $\lambda \in \text{pcf}(\mathfrak{c} \setminus \mathfrak{c}')$ (by 2.8(0)) and similarly $\mathfrak{c} \setminus \mathfrak{c}' \subseteq \text{pcf}_{\theta\text{-complete}}(\mathfrak{a} \setminus \mathfrak{a}')$ hence by parts (2),(3) of Claim 2.8 we have $\lambda \in \text{pcf}_{\theta\text{-complete}}(\mathfrak{a} \setminus \mathfrak{a}')$, contradiction).

Also without loss of generality $\lambda \notin \mathfrak{c}$. Let $\kappa, \sigma, \bar{N}, \langle i_\alpha = i(\alpha) : \alpha \leq \sigma \rangle, \bar{\mathfrak{a}} = \langle \mathfrak{a}_i : i \leq \sigma \rangle$ be as in 2.4 with $\mathfrak{a} \in N_0, \mathfrak{c} \in N_0, \lambda \in N_0, \sigma = |\mathfrak{a}|^+, \kappa = |\mathfrak{a}|^{+3} < \text{Min}(\mathfrak{a})$. We choose by induction on $\epsilon < |\mathfrak{a}|^+, \lambda_\epsilon, \mathfrak{d}_\epsilon$ such that:

- (a) " $\lambda_\epsilon \in \mathfrak{a}_{\omega^2\epsilon+\omega+1}, \mathfrak{d}_\epsilon \in N_{i(\omega^2\epsilon+\omega+1)},$
- (b) $\lambda_\epsilon \in \mathfrak{c},$
- (c) $\mathfrak{d}_\epsilon \subseteq \mathfrak{a}_{\omega^2\epsilon+\omega+1} \cap \text{pcf}_{\theta\text{-complete}}(\{\lambda_\zeta : \zeta < \epsilon\}),$
- (d) $|\mathfrak{d}_\epsilon| < \theta,$
- (e) $\{\lambda_\zeta : \zeta < \epsilon\} \subseteq \bigcup_{\theta \in \mathfrak{d}_\epsilon} \mathfrak{b}_\theta^{\omega^2\epsilon+\omega+1}[\bar{\mathfrak{a}}],$
- (f) $\lambda_\epsilon \notin \text{pcf}_{\theta\text{-complete}}(\bigcup_{\theta \in \mathfrak{d}_\epsilon} \mathfrak{b}_\theta^{\omega^2\epsilon+\omega+1}[\bar{\mathfrak{a}}]).$

For every $\epsilon < |\mathfrak{a}|^+$ we first choose \mathfrak{d}_ϵ as the $<_\chi^*$ -first element satisfying (c) + (d) + (e) and then if possible λ_ϵ as the $<_\chi^*$ -first element satisfying (b) + (f). It is easy to check the requirements and in fact $\langle \lambda_\zeta : \zeta < \epsilon \rangle \in N_{\omega^2\epsilon+1}, \langle \mathfrak{d}_\zeta : \zeta < \epsilon \rangle \in N_{\omega^2\epsilon+1}$ (so clause (a) will hold). But why can we choose at all? Now $\lambda \notin \text{pcf}_{\theta\text{-complete}}\{\lambda_\zeta : \zeta < \epsilon\}$ as $\mathfrak{a}, \mathfrak{c}, \lambda$ form a counterexample with λ minimal and $\epsilon < |\mathfrak{a}|^+$ (by 2.8(3)). As $\lambda = \max \text{pcf}(\mathfrak{a})$ necessarily $\text{pcf}_{\theta\text{-complete}}(\{\lambda_\zeta : \zeta < \epsilon\}) \subseteq \lambda$ hence $\mathfrak{d}_\epsilon \subseteq \lambda$ (by clause (c)). By part (0) of Claim 2.8 (and clause (a)) we know:

$$\begin{aligned} \text{pcf}_{\theta\text{-complete}}[\bigcup_{\mu \in \mathfrak{d}_\epsilon} \mathfrak{b}_\mu^{\omega^2\epsilon+\omega+1}[\bar{\mathfrak{a}}]] &= \bigcup_{\mu \in \mathfrak{d}_\epsilon} \text{pcf}_{\theta\text{-complete}}[\mathfrak{b}_\mu^{\omega^2\epsilon+\omega+1}[\bar{\mathfrak{a}}]] \\ &\subseteq \bigcup_{\mu \in \mathfrak{d}_\epsilon} (\mu + 1) \subseteq \lambda \end{aligned}$$

(note $\mu = \max \text{pcf}(\mathfrak{b}_\mu^\beta[\bar{\mathfrak{a}}])$). So $\lambda \notin \text{pcf}_{\theta\text{-complete}}(\bigcup_{\mu \in \mathfrak{d}_\epsilon} \mathfrak{b}_\mu^{\omega^2\epsilon+\omega+1}[\bar{\mathfrak{a}}])$ hence by part (0) of Claim 2.8 $\mathfrak{c} \not\subseteq \bigcup_{\mu \in \mathfrak{d}_\epsilon} \mathfrak{b}_\mu^{\omega^2\epsilon+\omega+1}[\bar{\mathfrak{a}}]$ so λ_ϵ exists. Now \mathfrak{d}_ϵ exists by 2.4 clause (h)⁺.

Now clearly $\left\langle \mathfrak{a} \cap \bigcup_{\mu \in \mathfrak{d}_\epsilon} \mathfrak{b}_\mu^{\omega^2\epsilon+\omega+1}[\bar{\mathfrak{a}}] : \epsilon < |\mathfrak{a}|^+ \right\rangle$ is non-decreasing (as in the earlier proof) hence eventually constant, say for $\epsilon \geq \epsilon(*)$ (where $\epsilon(*) < |\mathfrak{a}|^+$).

But

- (α) $\lambda_\epsilon \in \bigcup_{\mu \in \mathfrak{d}_{\epsilon+1}} \mathfrak{b}_\mu^{\omega^2\epsilon+\omega+1}[\bar{\mathfrak{a}}]$ [clause (e) in the choice of $\lambda_\epsilon, \mathfrak{d}_\epsilon$],
- (β) $\mathfrak{b}_{\lambda_\epsilon}^{\omega^2\epsilon+\omega+1}[\bar{\mathfrak{a}}] \subseteq \bigcup_{\mu \in \mathfrak{d}_{\epsilon+1}} \mathfrak{b}_\mu^{\omega^2\epsilon+\omega+1}[\bar{\mathfrak{a}}]$ [by clause (f) of 2.4 and (α) alone],
- (γ) $\lambda_\epsilon \in \text{pcf}_{\theta\text{-complete}}(\mathfrak{a})$ [as $\lambda_\epsilon \in \mathfrak{c}$ and a hypothesis],
- (δ) $\lambda_\epsilon \in \text{pcf}_{\theta\text{-complete}}(\mathfrak{b}_{\lambda_\epsilon}^{\omega^2\epsilon+\omega+1}[\bar{\mathfrak{a}}])$ [by (γ) above and clause (e) of 2.4],
- (ϵ) $\lambda_\epsilon \notin \text{pcf}(\mathfrak{a} \setminus \mathfrak{b}_{\lambda_\epsilon}^{\omega^2\epsilon+\omega+1})$,
- (ζ) $\lambda_\epsilon \in \text{pcf}_{\theta\text{-complete}}(\mathfrak{a} \cap \bigcup_{\mu \in \mathfrak{d}_{\epsilon+1}} \mathfrak{b}_\mu^{\omega^2\epsilon+\omega+1}[\bar{\mathfrak{a}}])$ [by (δ) + (ϵ) + (β)].

But for $\epsilon = \epsilon(*)$, the statement (ζ) contradicts the choice of $\epsilon(*)$ and clause (f) above. $\square_{2.9}$

§ 3.

Definition 3.1. 1) For J an ideal on κ (or any set, $\text{Dom}(J)$ -does not matter) and singular μ (usually $\text{cf}(\mu) \leq \kappa$, otherwise the result is 0)

(a) we define $\text{pp}_J(\mu)$ as

$$\sup\{\text{tcf}(\prod_{i<\kappa} \lambda_i, <_J) : \lambda_i \in \text{Reg} \cap \mu \setminus \kappa^+ \text{ for } i < \kappa \\ \text{and } \mu = \lim_J \langle \lambda_i : i < \kappa \rangle, \text{ see 3.2(1) and} \\ (\prod_{i<\kappa} \lambda_i, <_J) \text{ has true cofinality}\}$$

(b) we define $\text{pp}_J^+(\mu)$ as

$$\sup\{(\text{tcf}(\prod_{i<\kappa} \lambda_i, <_J))^+ : \lambda_i \in \text{Reg} \cap \mu \setminus \kappa^+ \text{ for } i < \kappa \\ \text{and } \mu = \lim_J \langle \lambda_i : i < \kappa \rangle, \text{ see 3.2(1) below and} \\ (\prod_{i<\kappa} \lambda_i, <_J) \text{ has true cofinality}\}.$$

2) For \mathbf{J} a family of ideals on (usually but not necessarily on the same set) and singular μ let $\text{pp}_{\mathbf{J}}(\mu) = \sup\{\text{pp}_J(\mu) : J \in \mathbf{J}\}$ and $\text{pp}_{\mathbf{J}}^+(\mu) = \sup\{\text{pp}_J^+(\mu) : J \in \mathbf{J}\}$.

3) For a set \mathfrak{a} of regular cardinals let $\text{pcf}_J(\mathfrak{a}) = \{\text{tcf}(\prod_{t \in \text{Dom}(J)} \lambda_t, <_J) : \lambda_t \in \mathfrak{a} \text{ for } t \in \text{Dom}(J)\}$; similarly $\text{pcf}_{\mathbf{J}}(\mathfrak{a})$.

Remark 3.2. 1) Recall that $\mu = \lim_J \langle \lambda_t : t \in \text{Dom}(J) \rangle$, where J is an ideal on $\text{Dom}(J)$ mean that for every $\mu_1 < \mu$ the set $\{t \in \text{Dom}(J) : \lambda_t \notin (\mu_1, \mu]\}$ belongs to J .

2) On $\text{pcf}_J(\mathfrak{a})$: check consistency of notation by [Sh:g].

Observation 3.3. 1) For μ, J as in clause (a) 3.1, the following are equivalent

- (a) $\text{pp}_J(\mu) > 0$
- (b) the sup is on a non-empty set
- (c) there is an increasing sequence of length $\text{cf}(\mu)$ of member of J whose union is κ
- (d) $\text{pp}_J(\mu) > \mu$
- (e) every cardinal appearing in the sup is regular $> \mu$ and the set of those appearing is $\text{Reg} \cap [\mu^+, \text{pp}_J^+(\mu))$ and is non-empty.

Definition 3.4. 1) Assume J is an ideal on $\kappa, \sigma = \text{cf}(\sigma) \leq \kappa, f \in {}^\kappa \text{Ord}$ then we let

$$\mathbf{W}_{J,\sigma}(f^*, < \mu) = \text{Min}\{|\mathcal{P}| : \mathcal{P} \text{ is a family of subsets of } \sup \text{Rang}(f^*) + 1 \\ \text{each of cardinality } < \mu \text{ and for every } f \leq f^*, \\ \text{Rang}(f) \text{ is the union of } < \sigma \\ \text{sets of the form} \\ \{i < \kappa : f(i) \in A\}, A \in \mathcal{P}\}.$$

2) If f^* is constantly λ we write λ if $\mu = \lambda$ we can omit $< \mu$.

Remark 3.5. 1) See $\text{cov}(\lambda, \mu, \theta, \sigma) = \mathbf{W}_{[\theta]^{<\sigma}, \sigma}(\langle \lambda : i < \theta \rangle, \mu)$.

2) On the case of normal ideals, i.e. prc see [Sh:410, §1] and more generally prd see [Sh:410].

We may use several families of ideals.

Definition 3.6. Let

- (a) $\text{com}_{\theta, \sigma} = \{J : J \text{ is a } \sigma\text{-complete ideal on } \theta\}$
- (b) $\text{nor}_\kappa = \{J : J \text{ a normal ideal on } \kappa\}$
- (c) $\text{com}_{I, \sigma} = \{J : J \text{ is a } \sigma\text{-complete ideal on } \text{Dom}(I) \text{ extending the ideal } I\}$
- (d) $\text{nor}_I = \{J : J \text{ is a normal ideal on } \text{Dom}(I) \text{ extending the ideal } I\}$.

Claim 3.7. *The $(< \aleph_1)$ -covering lemma.*

Assume $\aleph_1 \leq \sigma \leq \text{cf}(\mu) \leq \kappa < \mu$ and I is a σ -complete ideal on κ .

Then

- (a) $\mathbf{W}_{I, \sigma}(\mu) = \text{pp}_{\text{com}_\sigma(I)}(\mu)$
- (b) *except when $\otimes_{\mu, I, \sigma}$ below holds, we can strengthen the equality in clause (a) to: i.e., if $\text{pp}_{\text{com}_\sigma(I)}$ is a regular cardinal (so $> \mu$) then the sup in 3.1(1) is obtained*
- $\otimes_{\mu, I, \sigma}$ (a) $\lambda =: \text{pp}_{\text{com}_\sigma(I)}(\mu)$ is (weakly) inaccessible, the sup is not obtained and for some set $\mathfrak{a} \subseteq \text{Reg} \cap \mu$, $|\mathfrak{a}| + \kappa < \text{Min}(\mathfrak{a})$ and $\lambda = \sup(\text{pcf}_{I, \sigma}(\mathfrak{a}))$; recalling $\text{pcf}_{\text{com}_\sigma(I)}(\mathfrak{a}) = \{\prod_{i < \kappa} \lambda_i, <_J : J \in \text{com}_\sigma(I), \lambda_i \in \mathfrak{a} \text{ for } i < \kappa\}$.

Remark 3.8. 1) This is [Sh:513, 6.13].

In a reasonable case the result $\text{cov}(|\mathfrak{a}|, \kappa^+, \kappa^+, \sigma)$.

Conclusion 3.9. *In 3.7 if $\kappa < \mu_* \leq \mu$ then*

- (a) $\mathbf{W}_{I, \sigma}(\mu, < \mu_*) = \sup\{\text{pp}_{\text{com}_\sigma(I)}(\mu)' : \mu_* \leq \mu' \leq \mu, \text{cf}(\mu') \leq \kappa\}$
- (b) *if in (a) the left side is a regular cardinal then the sup is obtained for some sequence $\langle \lambda_i : i < \kappa \rangle$ of regular cardinality and $J \in \text{com}_\sigma(I)$ such that $\lim_J \langle \lambda_i : i < \kappa \rangle$ is well defined and $\in [\mu_*, \mu]$ except possibly when*
- $\otimes_{\mu, I, \sigma, \mu_*}$ *as in $\otimes_{\mu, I, \sigma}$ above but $|\mathfrak{a}| < \mu_*$.*

Proof. The inequality \geq :

So assume J is a σ -complete ideal on κ extending I , $\lambda_i \in \text{Reg} \cap \mu \setminus \kappa^+$ and $\mu = \lim_J (\langle \lambda_i : i < \kappa \rangle)$ and $\lambda = \text{tcf}(\prod_{i < \kappa} \lambda_i, <_J)$ is well defined and we shall note that

$\mathbf{W}_{I, \sigma}(\mu) \geq \lambda$, this clearly suffices, and let $\langle f_\alpha : \alpha < \lambda \rangle$ be $<_J$ -increasing cofinal in $(\prod_{i < \kappa} \lambda_i, <_J)$. Now let $|\mathcal{P}| < \lambda$, \mathcal{P} be a family of sets of ordinals each of cardinality $< \mu$. For each $u \in \mathcal{P}$ let $g_u \in \prod_{i < \kappa} \lambda_i$ be defined by $g_u(i) = \sup(u \cap \lambda_i)$ if $|u| < \lambda_i$

and $g_u(i) = 0$ otherwise.

Hence for some $\alpha(u) < \lambda$, $g_u <_J f_{\alpha(u)}$ and so $\alpha(*) = \cup\{\alpha(u) + 1 : u \in \mathcal{P}\} < \lambda$ and $f_{\alpha(*)}$ exemplifies the failure of \mathcal{P} to exemplify $\lambda > \mathbf{W}_{I, \sigma}(\mu)$.

The inequality \leq :

Assume that λ is regular $\geq \text{pp}_{I, \sigma}^+(\mu)$ and we shall prove that $\mathbf{W}_{I, \sigma}(\mu) < \lambda$, this clearly suffices. Let χ be large enough, and \mathfrak{B} be an elementary submodel of $(\mathcal{H}(\chi), \in, <_\chi^*)$ of cardinality $< \lambda$ such that $\{I, \sigma, \mu, \lambda\} \subseteq \mathfrak{B}$ and $\lambda \cap \mathfrak{B}$ is an ordinal

which we shall call $\delta_{\mathfrak{B}}$. Let $\mathcal{P} =: [\mu]^{<\mu} \cap \mathfrak{B}$ so $|\mathcal{P}| < \lambda$. Hence it is enough to prove that $\mathbf{W}_{I,\sigma}(\mu) \leq |\mathcal{P}|$ and for this it is enough to prove that \mathcal{P} is as required in Definition 3.3(1). Let $\bar{e} = \langle e_\alpha : \alpha < \mu \rangle \in \mathfrak{B}$ be such that e_α is a club of α of order type $\text{cf}(\alpha)$ so $e_{\alpha+1} = \{\alpha\}$, $e_0 = \emptyset$.

So let $f_* \in {}^\kappa \mu$ and let $\langle \mu_\varepsilon : \varepsilon < \text{cf}(\mu) \rangle \in \mathfrak{B}$ be an increasing continuous sequence of cardinals from (κ, μ) with limit μ . Now by induction on $n < \omega$ we choose $\varepsilon_n, A_n, g_n, \mathcal{T}_n, \bar{S}_n, \bar{B}_n$ such that

- \circledast_n (A)(a) $A_n \in [\mu]^{<\kappa}$, $A_0 = \{\mu_\varepsilon : \varepsilon < \text{cf}(\mu)\}$
- (b) g_n is a function from κ to A_n
- (c) $f_* \leq g_n$
- (d) if $n = m + 1$ and $i < \kappa$ then $g_m(i) > f_*(i) \Rightarrow g_n(i) > g_m(i)$
- (e) $\mathcal{T}_n \subseteq {}^n \sigma$ has cardinality $< \sigma$
- (f) $\mathcal{T}_0 = \{<>\}$
- (g) if $n = m + 1$ and $\eta \in \mathcal{T}_n$ then $\eta \upharpoonright m \in \mathcal{T}_m$
- (h) $\bar{S}_n = \langle S_\eta : \eta \in \mathcal{T}_n \rangle$
- (i) $\bar{B}_n = \langle B_\eta : \eta \in \mathcal{T}_n \rangle$
- (j) $\varepsilon_n < \text{cf}(\mu)$ and $n = m + 1 \Rightarrow \varepsilon_n \geq \varepsilon_m$
- (B) for each $\eta \in \mathcal{T}_n$:
 - (a) $S_\eta \subseteq \kappa$, $S_\eta \notin \mathcal{T}_n$
 - (b) if $n = m + 1$ then $S_{\eta \upharpoonright m} \supseteq S_\eta$
 - (c) $B_\eta \in \mathfrak{B}$ is a subset of μ of cardinality $< \mu_{\varepsilon(n)}$
 - (d) $\{g_n(i) : i \in S_\eta\}$ is included in B_η
- (C)(a) if $n = m + 1$ and $\eta \in \mathcal{T}_m$ then the set

$$S_\eta^* := \{i \in S_\eta : g_m(i) > f_*(i)\} \setminus \cup \{S_{\eta \hat{\cdot} \langle j \rangle} : \eta \hat{\cdot} \langle j \rangle \in \mathcal{T}_n\}$$
 belongs to I .

It is enough to Carry the definition:

Why? As then $\{B_\eta : \eta \in \mathcal{T}_n \text{ for some } n < \omega\}$ is a family of members of \mathcal{P} (by (B)(c)), its cardinality is $< \sigma$ (as $\sigma = \text{cf}(\sigma) > \aleph_0$ and for each $n < \omega$, $|\mathcal{T}_n| < \sigma$ by (A)(e)).

Similarly as I is σ -complete the set $S^* = \cup \{S_\eta^* : \eta \in \mathcal{T}_n \text{ for some } n < \omega\}$ belongs to I . Now for every $i \in \kappa \setminus S^*$, we try to choose $\eta_n \in \mathcal{T}_n$ by induction on $n < \omega$ such that $i \in S_{\eta_n}$ and $n = m + 1 \Rightarrow \eta_m = \eta_n \upharpoonright m$ and $g_m(i) > f_*(i)$. For $n = 0$ let $\eta = <>$ so $i \in \kappa = A_0$. For $n = m + 1$, as $i \notin S_{\eta_m}^*$, see (C)(a) clearly η_m as required exists. Now if $n = m + 1$ again as $i \notin S_{\eta_m}^*$ we get $g_m(i) > f_*(i)$ and by (A)(d) we have $g_m(i) > g_n(i)$. But there is no decreasing ω -sequence of ordinals. So for some m , $g_m(i) \leq f_*(i)$ so by (A)(c), $g_m(i) = f_*(i)$ but $g_n(i) \in B_{\eta_n}$.

Carrying the induction:

Case $n = 0$:

Let $\mathcal{T}_0 = \{<>\}$, $A_{<>} = \{\mu_\varepsilon : \varepsilon < \text{cf}(\mu)\}$ which has cardinality $\leq \kappa$ as $\text{cf}(\mu) \leq \kappa$ by assumption. Further, let g_0 be defined as the function with domain κ and $g_0(i) = \min\{\mu_\varepsilon : \mu_\varepsilon > f_*(i)\}$, let $S_{<>} = \kappa$ and $B_{<>} = A_0$ which $\in \mathfrak{B}$ as $\langle \mu_\varepsilon : \varepsilon < \text{cf}(\mu) \rangle \in \mathfrak{B}$ (and has cardinality $|A_0| = \text{cf}(\mu) \leq \kappa$).

Case $n = m + 1$:

Let $\eta \in \mathcal{T}_m$ and define $S'_\eta = \{i \in S_\eta : g_n(i) > f_*(i)\}$. If $S'_\eta \in I$ then we decide that $j < n \Rightarrow \eta \smallfrown \langle j \rangle \notin \mathcal{T}_n$, so we have nothing more to do so assume $S'_\eta \notin I$.

Let $\mathfrak{a}_\eta = \{\text{cf}(\alpha) : \alpha \in B_\eta \text{ and } \text{cf}(\alpha) > |B_\eta| + \kappa\}$ and let

$$\mathfrak{c}_\eta = \{\text{tcf}(\prod_{i \in S'_\eta} \text{cf}(g_n(i)), <_J) : J \text{ is an } \sigma\text{-complete ideal on}$$

$$S'_\eta \text{ extending } I \upharpoonright S'_\eta \text{ such that } \mu = \lim_J \langle \text{cf}(g_n(i)) : i \in S'_\eta \rangle \\ \text{and } \prod_{i \in S'_\eta} \text{cf}(g_n(i)), <_J \text{ has true cofinality}\}$$

Clearly $\kappa + |\mathfrak{a}_\eta| < \min(\mathfrak{a}_\eta)$ and $\mathfrak{c}_\eta \subseteq \text{pcf}_{I,\sigma}(\mathfrak{a}_\eta) \subseteq \lambda \cap \text{Reg}$ and by $\neg \otimes_{\mu, I, \sigma}$ we know that $\text{pcf}_{I,\sigma}(\mathfrak{a}_\eta)$ is a bounded subset of λ . But $B_\eta \in \mathfrak{B}$ hence $\mathfrak{a}_\eta \in \mathfrak{B}$ hence $\text{pcf}_{I,\sigma}(\mathfrak{a}_\eta) \in \mathfrak{B}$ so as $\mathfrak{B} \cap \lambda = \delta_{\mathfrak{B}} < \lambda$, clearly $\text{pcf}_{I,\sigma}(\mathfrak{a}_\eta) \subseteq \mathfrak{B}$ hence $\theta \in \mathfrak{c}_\eta \Rightarrow \theta < \delta_{\mathfrak{B}}$. Using pcf basic properties let $J_{\eta,\lambda}$ be the σ -complete ideal on \mathfrak{a}_η generated by $J_{=\lambda}[\mathfrak{a}_\eta]$ and so $\bar{\mathfrak{a}}_\eta, J_{\eta,\lambda} \in \mathfrak{B}$ and there is a $<_{J_{\eta,\lambda}}$ -increasing cofinal sequence $\bar{f}_{\eta,\lambda} = \langle f_{\eta,\lambda,\zeta} : \zeta < \lambda \rangle$ of members of $\Pi \mathfrak{a}_\eta$ such that $f_{\eta,\lambda,\zeta}$ is the $<_{J_{\eta,\lambda}}$ -e.u.b. of $\bar{f}_{\eta,\lambda} \upharpoonright \zeta$ when there is such $<_{J_{\eta,\lambda}}$ -e.u.b. Without loss of generality $\bar{f}_{\eta,\lambda} \in \mathfrak{B}$ hence $\{f_{\eta,\lambda,\zeta} : \zeta < \lambda\} \subseteq \mathfrak{B}$.

Let $\mathfrak{a}_m = \cup\{\mathfrak{a}_\eta : \eta \in \mathcal{T}_m\}$ and define a $h_m \in \Pi \mathfrak{a}_m$ by $h_m(\theta) = \sup\{\text{otp}(e_{g_m(i)} \cap f_*(i)) : i < \kappa \text{ and } f_*(i) < g_m(i)\}$. Clearly it is $< \theta$ as $\theta = \text{cf}(\theta) > \mu_{\varepsilon(m)} \geq |B_\eta| + \kappa$ when $\theta \in \mathfrak{a}_\eta$. For each $\eta \in \mathcal{T}_m$ and $\lambda \in \mathfrak{c}_\eta$ let $\zeta_{\eta,\lambda} < \lambda$ be such that $h_m \upharpoonright \mathfrak{a}_\eta < f_{\eta,\lambda,\zeta_{\eta,\lambda}} \text{ mod } J_{\eta,\lambda}$, and let

$$S_{\eta,\lambda}^1 = \{i \in S_\eta : h_m(\text{cf}(g_i(\theta))) < f_{\eta,\lambda,\zeta_{\eta,\lambda}}(\text{cf}(g_m(i)))\}$$

- ⊙ for some subset \mathfrak{c}'_η of \mathfrak{c}_η of cardinality $< \sigma$ the set $\{i \in S_\eta : i \notin S_{\eta,\lambda}^1 \text{ for every } \lambda \in \mathfrak{c}'_\eta\}$ belongs to I .

[Why? Otherwise, let J be the σ -complete ideal on S_η generated by $I \cup \{S_{\eta,\lambda}^1 : \lambda \in \mathfrak{c}_\eta\}$, so $\kappa \notin J$ hence for some $S^* \in J^+$ we know that $(\prod_{i \in S^*} \text{cf}(g_m(i)), <_{J \upharpoonright S^*})$ has true cofinality, call it λ^* . Necessarily $\lambda^* \in \mathfrak{c}_\eta$ and easily get a contradiction.]

Case A: $|\cup\{\mathfrak{c}_\eta : \eta \in \mathcal{T}_m\}| < \mu$.

Let $\langle \lambda_{\eta,j} : j < j_\eta \rangle$ list \mathfrak{c}'_η . Let $\mathfrak{a}'_n = \mathfrak{a}_n \setminus \bigcup_{\eta} \mathfrak{c}_\eta$. Now by induction on $k < \omega$ we choose $h_{n,k}, \zeta_{\eta,j,k}$ for $j < j_\eta, \eta \in \mathcal{T}_m$ such that

- ⊗ (a) $h_{m,k} \in \Pi \mathfrak{a}'_m$
- (b) $h_{m,k} < h_{m,k+1}$
- (c) $h_{m,0} = h_m$
- (d) $\zeta_{\eta,j,k} < \lambda_{\eta,j}$
- (e) $\zeta_{\eta,j,k} < \zeta_{\eta,j,k+1}$
- (f) $\zeta_{\eta,j,0} = \zeta_{\eta,j}$
- (g) $h_{m,k+1}(\theta) = \sup[\{f_{\eta,\lambda_{\eta,j},\zeta_{\eta,j,k}}(\theta) : \eta \in \mathcal{T}_n, \theta \in \mathfrak{a}_\eta\} \cup \{h_{m,k}(\theta)\}]$
- (h) $\zeta_{\eta,j,k+1} = \text{Min}\{\zeta < \lambda_{\eta,j} : \zeta > \zeta_{\eta,j,k} \text{ and } h_{m,k+1} \upharpoonright \mathfrak{a}_\eta < f_{\eta,\lambda_{\eta,j},\zeta} \text{ mod } J_{\eta,\lambda_{\eta,j}}\}$.

There is no problem to carry the induction. Let $h_{m,\omega} \in \Pi \mathbf{a}_m$ be defined by $h_{m,\omega}(\theta) = \cup \{h_{m,k}(\theta) : k < \omega\}$. Let $S'_{\eta,j} = \{i \in S_\eta : f_*(i) \text{ is } < \text{the } h_{m,\omega}(\text{cf}(g_m(i))\text{-ith member of } e_{g_m(i)})\}$.

Now

⊠ for some $\mathfrak{c}''_\eta \subseteq \mathfrak{c}_\eta, |\mathfrak{c}''_\eta| < \sigma$ for $\eta \in \mathcal{T}_m$ we have $S_n \setminus \cup \{S_{\eta,j} : \lambda_j \in \mathfrak{c}'_\eta\} \in I$.

Now continue. □_{3.7}

Case B: C not Case A.

Use §2.

* * *

Discussion 3.10. Lemma 3.7 leaves us in a strange situation: clause (a) is fine, but concerning the exception in clause (b); it may well be impossible and $\text{pcf}(\mathbf{a})$ is always not “so large”. We do not know this, we try to clarify the case for reasonable \mathbf{J}_i , i.e., closed under products of two.

Observation 3.11. 1) *There is $\mu_* < \mu$ such that $(\forall \mu')(\mu_* < \mu' \leq \mu \wedge \text{cf}(\mu') \leq \kappa < \mu') \Rightarrow \text{pp}_{\mathbf{J}}^+(\mu') \leq \text{pp}_{\mathbf{J}}^+(\mu)$ when:*

- ⊗ (a) $\text{cf}(\mu) \leq \kappa < \mu$
- (b) \mathbf{J} is a set of σ -complete ideals
- (c) $J \in \mathbf{J} \Rightarrow |\text{Dom}(J)| \leq \kappa$
- (d) if $J_\varepsilon \in \mathbf{J}$ for $\varepsilon < \text{cf}(\mu)$ then for some σ -complete ideal I on $\text{cf}(\mu)$, the ideal $J = \Sigma_I \langle J_\varepsilon : \varepsilon < \text{cf}(\mu) \rangle$ belongs to \mathbf{J} (or is just \leq_{RK} from some $J' \in \mathbf{J}$).

Proof. 3.11 Let $\Lambda = \{\mu' : \mu' \text{ is a cardinal } < \mu \text{ but } > \kappa, \text{ of cofinality } \leq \kappa \text{ such that } \text{pp}_{\mathbf{J}}^+(\mu') > \text{pp}_{\mathbf{J}}(\mu)\}$, and assume toward contradiction that $\mu = \sup(\Lambda)$. So we can choose an increasing sequence $\langle \mu_\varepsilon : \varepsilon < \text{cf}(\mu) \rangle$ of members of Λ with limit μ . For each $\varepsilon < \text{cf}(\mu)$ let $J_\varepsilon \in \mathbf{J}$ witnesses $\mu_\varepsilon \in \Lambda$. Without loss of generality $\kappa_\varepsilon = \text{Dom}(J_\varepsilon) \leq \kappa$ so we can find $\langle \lambda_{\varepsilon,i} : i < \kappa_\varepsilon \rangle$ witnessing this. In particular $(\prod_{i < \kappa_\varepsilon} \lambda_{\varepsilon,i}, <_{J_\varepsilon})$ has true cofinality $\lambda_\varepsilon = \text{cf}(\lambda_\varepsilon) \geq \text{pp}_{\mathbf{J}}^+(\mu)$. Let I, J be as in clause (d) of ⊗. □

* *

A dual kind of measure to Definition 3.1 is

Definition 3.12. 1) Assume J is an ideal say on κ and $f^* : \kappa \rightarrow \text{Ord}$ and μ cardinal. Then $\mathbf{U}_J(f^*, < \mu) = \text{Min}\{|\mathcal{P}| : \mathcal{P} \text{ a family of subsets of } \sup \text{Rang}(f) + 1 \text{ each of cardinality } < \mu \text{ such that for every } f \leq f^* \text{ (i.e., } f \in \prod_{i < \kappa} (f^*(i) + 1)) \text{ there is}$

$A \in \mathcal{P}$ such that $\{i < \kappa : f(i) \in A\} \notin J\}$.

2) If above we write \mathbf{J} instead of J this means \mathbf{J} is a family of ideals on κ and the \mathcal{P} should serve all the $J \in \mathbf{J}$ simultaneously.

Claim 3.13. *We have $\mathbf{U}_{J_{\kappa}^{\text{bd}}}(\mu, < \mu) = \lambda_*$ if we assume*

- ⊗ (a) $\mu > \kappa = \text{cf}(\mu) > \aleph_0$

- (b) $([\kappa]^\kappa, \supseteq)$ satisfies the μ -c.c. or just μ^+ -c.c. which means that:
if $\mathcal{A} \subseteq [\kappa]^\kappa$ and $A \neq B \in \mathcal{A} \Rightarrow |A \cap B| < \kappa$ then $|\mathcal{A}| \leq \mu$
- (c) $\lambda_* = \text{pp}_{J_\kappa^{\text{bd}}}(\mu) = \sup\{\text{tcf}(\prod_{i < \kappa} \lambda_i, <_{J_\kappa^{\text{bd}}}) : \lambda_i < \mu \text{ is increasing with}$
 $\text{limit } \mu \text{ and } (\prod_{i < \kappa} \lambda_i, <_{J_\kappa^{\text{bd}}}) \text{ has true cofinality}\}.$

Claim 3.14. We can in 3.13 replace J_κ^{bd} by any \aleph_1 -complete filter J (?) on κ (so (b) becomes “ (J^+, \supseteq) satisfies the μ^+ -c.c.”)

Remark 3.15. If in clause (b) of \otimes of 3.13, we use the μ -c.c. the proof is simpler, using $\mathcal{T}_n \subseteq {}^n(\mu_{\varepsilon_n}), \varepsilon_n \leq \varepsilon_{n+1}$.

Proof. Let

- (*) (a) $\bar{\mu} = \langle \mu_i : i < \kappa \rangle$ is an increasing continuous sequence of singular cardinals $> \kappa$ with limit μ .

Let χ be large enough, $<_\chi^*$ a well ordering of $(\mathcal{H}(\chi), \in)$ and \mathcal{B} an elementary submodel of $(\mathcal{H}(\chi), \in, <_\chi^*)$ of cardinality λ_* such that $\lambda_* + 1 \subseteq gB$ and $\bar{\mu} \in \mathfrak{B}$ and let $\mathcal{A} = [\mu]^{<\mu} \cap \mathfrak{B}$.

So \mathcal{A} is a family of sets of the right form and has cardinality $\leq \lambda_*$. It remains to prove the major point: assume S is an unbounded subset of κ , $f^* \in \prod_{i \in S} [\mu_i, \mu_{i+1}]$

we should prove that $(\exists A \in \mathcal{A})(\exists^\kappa i \in S)(f(i) \in A)$.

Let $\bar{e} = \langle e_\alpha : \alpha < \mu \rangle \in \mathfrak{B}$ be such that e_α is a club of α of order type $\text{cf}(\alpha)$ so $e_{\alpha+1} = \{\alpha\}, e_0 = \emptyset$. Let $\langle \beta_{\alpha, \varepsilon} : \varepsilon < \text{cf}(\alpha) \rangle$ be an increasing enumeration of e_α .

We choose $\varepsilon_n, g_n, A_n, I_n, \langle S_\eta, B_\eta : \eta \in \mathcal{T}_n \rangle$ such that

- \otimes_n (A) (a) $\mathcal{T}_n \subseteq {}^n\mu, \mathcal{T}_0 = \{<\>\}, [n = m + 1 \wedge \eta \in \mathcal{T}_n \Rightarrow \eta \upharpoonright m \in \mathcal{T}_m]$
- (b) $A_n \subseteq \mu$ has cardinality $\leq \kappa$
- (c) $g_n : \kappa \rightarrow A_n$
- (d) $i < \kappa \Rightarrow f^*(i) \leq g_n(i)$
- (e) $n = m + 1 \Rightarrow g_n \leq g_m$
- (f) $\varepsilon_n < \kappa$ and $n = m + 1 \Rightarrow \varepsilon_m < \varepsilon_n$
- (g) if $n = m + 1, i \in (\varepsilon_n, \kappa)$ and $g_m(i) > f^*(i)$ then $g_m(i) > g_n(i)$
- (B) for $\eta \in \mathcal{T}_n$
 - (a) $S_\eta \subseteq \kappa$ has cardinality κ
 - (b) $S_\eta \in [\kappa]^\kappa$ and $\nu \triangleleft \eta \Rightarrow S_\eta \subseteq S_\nu$
 - (c) $B_\eta \in \mathfrak{B}$ is a subset of μ of cardinality $< \mu_{\varepsilon(n)}$ where $\varepsilon(n) = \text{Min}\{\varepsilon < \kappa : \eta \in {}^n(\mu_\varepsilon) \text{ and } \varepsilon \geq \varepsilon_n\}$
 - (d) $\{g_n(i) : i \in S_\eta\} \subseteq B_\eta$.

For $n = 0$ let $\varepsilon_0 = 0, A_{<\>} = \{\mu_i : i < \kappa\}, \mathcal{T}_0 = \{<\>\}, S_{<\>} = \kappa, g_m$ is the function with domain κ such that $g_{<\>} = \text{Min}\{\alpha \in A_{<\>} : f^*(i) < \alpha\}$. Assume $n = m + 1$ and we have defined for m .

Let

$$\begin{aligned} \mathfrak{c}_n = \{ \theta : & \text{ there is an increasing sequence } \langle \lambda_i : i < \kappa \rangle \\ & \text{ of regular cardinals } \in (\kappa, \mu) \text{ with limit } \mu \text{ such that} \\ & \theta = \text{tcf}(\prod_{i < \kappa} \lambda_i, <_{J_\kappa^{\text{bd}}}) \text{ and} \\ & \{ \lambda_i : i < \kappa \} \subseteq \{ \text{cf}(\alpha) : \alpha \in A_m, \text{cf}(\alpha) > \kappa \}. \end{aligned}$$

Of course, $\mathfrak{c}_n \subseteq \text{Reg} \setminus \mu$. Now for each $\theta \in \mathfrak{c}_n$ let $\langle \lambda_i^\theta : i < \kappa \rangle$ exemplifies it so $\{ \{ \lambda_i^\theta : i < \kappa \} : \theta \in \mathfrak{c}_n \}$ is a family of subsets of $\{ \text{cf}(\alpha) : \alpha \in A_m, \text{cf}(\alpha) > \kappa \}$ each of cardinality κ and the intersection of any two has cardinality $< \kappa$.

As $|A_m| \leq \kappa$, by assumption (d) of the claim we know that $|\mathfrak{c}_n| \leq \mu$ and let $\langle \lambda_\beta : \beta \leq \mu \rangle$ list them.

For each $\eta \in \mathcal{T}_m$ and $\varepsilon < \kappa$ let

$$\mathfrak{a}_{\eta, \varepsilon} = \{ \text{cf}(\delta) : \delta \in B_\eta \text{ and } \text{cf}(\delta) > \mu_\varepsilon + |B_\eta| \}$$

so

$$|\mathfrak{a}_{\eta, \varepsilon}| \leq |B_\eta| < \min(\mathfrak{a}_\eta).$$

Let $W = \{ (\eta, \varepsilon, \beta) : \eta \in \mathcal{T}_m, \varepsilon < \kappa, \beta < \mu_\varepsilon \}$. Clearly $\mathfrak{a}_{\eta, \varepsilon} \in \mathfrak{B}, \lambda_\beta \in \mathfrak{B}$ hence $J_{\eta, \varepsilon, \beta} =$ the κ -complete ideal generated by $J_{=\lambda_\beta}[\mathfrak{a}_{\eta, \varepsilon}]$ belongs to \mathfrak{B} and some $<_{J_{\eta, \varepsilon, \beta}}$ -increasing and cofinal sequence $\langle f_{\eta, \varepsilon, \beta, \zeta} : \zeta < \lambda_\beta \rangle$ belongs to \mathfrak{B} and $f_{\eta, \varepsilon, \beta, \zeta}$ is an $<_{J_{\eta, \varepsilon, \beta}}$ -e.u.b. of $\langle f_{\eta, \varepsilon, \beta, \xi} : \xi < \zeta \rangle$ when there is one.

We now define a function h_m

$$\text{Dom}(h_m) = \mathfrak{a}_m^* = \cup \{ \mathfrak{a}_{\eta, \varepsilon} : \eta \in \mathcal{T}_m \text{ and } \varepsilon < \kappa \}$$

so

$$\theta \in \text{Dom}(h_m) \Rightarrow \kappa < \theta < \mu \wedge \theta \in \text{Reg}$$

(in fact we do not exclude the case $\mathfrak{a}_m^* = \text{Reg} \cap \mu \setminus \kappa^+$) and

$$h_m(\theta) = \sup \{ e_{g_n(i)} \cap f * (i) : i < \kappa \text{ and } \text{cf}(g_n(i)) = \theta \}.$$

As $\theta = \text{cf}(\theta) > \kappa$ clearly

$$\theta \in \text{Dom}(h_m) \Rightarrow h_m(\theta) < \theta.$$

We choose now by induction on $k < \omega, h_{m, k}, \langle \zeta_{\eta, \varepsilon, \beta}^k : (\eta, \varepsilon, \beta) \in W \rangle$ such that

- ⊠ (a) $h_{m, k} \in \Pi \mathfrak{a}_m^*$
- (b) $h_{m, 0} = h_m$
- (c) $h_{m, k} \leq h_{m, k+1}$
- (d) $\zeta_{\eta, \varepsilon, \beta}^k = \text{Min} \{ \zeta : h_{m, k} \restriction \mathfrak{a}_{\eta, \varepsilon} <_{J_{\eta, \varepsilon, \beta}} f_{\eta, \varepsilon, \beta, \zeta} \text{ and } \ell < k \Rightarrow \zeta_{\eta, \varepsilon, \beta}^\ell < \zeta \}$
- (e) $h_{m, k+1}(\theta) = \sup \{ h_{m, k}(\theta) \} \cup \{ f_{\eta, \beta, \varepsilon, \zeta_{\eta, \varepsilon, \eta}^k}^k(\theta) : \text{the triple } (\eta, \beta, \varepsilon) \in W \text{ satisfies } (\exists \varepsilon)(\beta < \mu_\varepsilon < \theta) \text{ and } \theta \in \mathfrak{a}_{\eta, \varepsilon} \}$.

Note that $h_{m,k+1}(\theta) < \theta$ as the sup is over a set of $< \theta$ ordinals.

So we have carried the definition, and let $h_{m,w}^* \in \Pi \mathfrak{a}_m$ be defined by $h_{m,\omega}(\theta) = \sup\{h_{m,k}(\theta) : k < \omega\}$ and $\zeta_{\eta,\varepsilon,\beta} = \zeta(\eta,\varepsilon,\beta) = \sup\{\zeta_{\eta,\varepsilon,\beta}^k : k < \omega\}$. Now for each $(\eta,\varepsilon,\beta) \in W$ we have $k < \omega \Rightarrow h_{m,k} \upharpoonright \mathfrak{a}_{\eta,\varepsilon} <_{J_{\eta,\varepsilon,\beta}} f_{\eta,\varepsilon,\beta,\zeta(\eta,\varepsilon,\beta)}^k < h_{m,k+1} \upharpoonright \mathfrak{a}_{\eta,\varepsilon}$. By the choice of $\bar{f}_{\eta,\varepsilon,\beta}$ as $J_{\eta,\varepsilon,\beta}$ is \aleph_1 -complete it follows that $h_{m,w} \upharpoonright \mathfrak{a}_{\eta,\varepsilon} = f_{\eta,\varepsilon,\beta,\zeta_{\eta,\varepsilon,\beta}} \bmod J_{\eta,\varepsilon,\beta}$.

Let

$$A_n = \{\alpha' : \text{for some } \alpha \in A_n, \text{cf}(\alpha) \in \mathfrak{a}_n \text{ and } \alpha' \text{ is the } h_{m,\omega}(\text{cf}(\alpha))\text{-th member of } e_\alpha\}.$$

$$g_n(i) \text{ is } \alpha' \text{ when } \alpha' \text{ is the } h_{m,\omega}(\text{cf}(g_m(i)))\text{-th member of } e_{g_m(i)} \text{ and zero otherwise.}$$

The main point is why $\sigma_n \in (\varepsilon_m, \kappa)$ exists.

To finish the induction step on n , let

$$B_{\eta,\varepsilon,\beta} = \text{Rang}(f_{\eta,\varepsilon,\eta,\zeta_{\eta,\varepsilon,\beta}})$$

$$B'_{\eta,\varepsilon} = B_{\eta,\varepsilon,\beta} \cup \{e_\alpha : \alpha \in B_{\eta,\varepsilon} \text{ and } \text{cf}(\alpha) \leq \mu_{\varepsilon(n)}\}$$

and we choose $\langle B_\rho : \rho \in \mathcal{T}_n, \rho \upharpoonright m \in B = \eta \text{ to list them enumerates } \{B_{\eta,\varepsilon,\beta} : \varepsilon, \beta\} \text{ are such that } (\eta, \varepsilon, \beta) \in W_m \cup \{B'_{\eta,\varepsilon}\} \text{ in a way consistent with the induction hypothesis.}$

Having carried the induction on n , note that

$$\otimes_1 \text{ for some } n, u_n = \{i < \kappa : f^*(i) = g_n(i)\} \in [\kappa]^\kappa$$

We now choose by induction on $m \leq n$ a sequence $\eta_m \in \mathcal{T}_m$ such that $\eta_0 = \langle \rangle$, $m = \ell + 1 \Rightarrow \eta_\ell \triangleleft \eta_m$ and $S_\eta \cap u_n \in [\kappa]^\kappa$. For $m = n$ by

$$\otimes(*) \ u' = u \cap S_{\eta_n} \in [\kappa]^\kappa \text{ and } \text{Rang}(f^* \cap u') \subseteq B_\eta \in \mathcal{P} \text{ so we are done.}$$

□

Discussion 3.16. 1) Can we consider “ $\mathfrak{c}([\mu]^\mu, \supseteq) \leq \mu^+$ ”? We should look again at §2.

2) More hopeful is to replace $\mathbf{U}_{J_\kappa^{\text{bd}}}(\mu)$ by $\mathbf{U}_{\text{non-stationary}_\kappa}(\mu)$.

3) By 3.11 and 3.12 we should have the prd version (for which \mathbf{J} and closure, see [Sh:410].

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